# EECS 16A Designing Information Devices and Systems I
# Summer 2020 Discussion 6A

**Reference: Inner products**

Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be vectors in real vector space $\mathbb{V}$. A mapping $\langle \cdot, \cdot \rangle$ is said to be an inner product on $\mathbb{V}$ if it satisfies the following three properties:

(a) Symmetry: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

(b) Linearity: $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$ and $\langle c\vec{x}, \vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$

(c) Positive-definiteness: $\langle \vec{x}, \vec{x} \rangle \geq 0$, with equality if and only if $\vec{x} = \vec{0}$.

We define the norm of $\vec{x}$ as $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$.

**Cross-correlation:**

The cross-correlation between two signals $r[n]$ and $s[n]$ is defined as follows:

$$\text{corr}_r(s)[k] = \sum_{i=-\infty}^{\infty} r[i]s[i-k].$$

## 1. Mechanical Inner Products

For the following pairs of vectors, find the Euclidean inner product $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y}$.

(a)

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

**Answer:** Recall that the inner product of two vectors $\vec{x}$ and $\vec{y}$ is $\vec{x}^T \vec{y}$, thus:

$$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 3 = 4$$

(b)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

**Answer:** When working with real numbers, the inner product is commutative. Thus, using our work from the previous part, the inner product of these two vectors is 4.

(c)

$$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

**Answer:**

$$\begin{bmatrix}1 & 0 & 3\end{bmatrix}\begin{bmatrix}-3\\2\\1\end{bmatrix} = -3+3 = 0$$

**2. Inner Product Properties**

Demonstrate the following properties of inner products for any vectors in $\mathbb{R}^2$, assuming we are working with the Euclidean inner product and norm.

(a) Symmetry

**Answer:** Let $x_i, y_i \in \mathbb{R}$, then

$$\begin{aligned}\left\langle \begin{bmatrix}x_1\\x_2\end{bmatrix}, \begin{bmatrix}y_1\\y_2\end{bmatrix}\right\rangle &= x_1 \cdot y_1 + x_2 \cdot y_2\\ &= y_1 \cdot x_1 + y_2 \cdot x_2\\ &= \left\langle \begin{bmatrix}y_1\\y_2\end{bmatrix}, \begin{bmatrix}x_1\\x_2\end{bmatrix}\right\rangle\end{aligned}$$

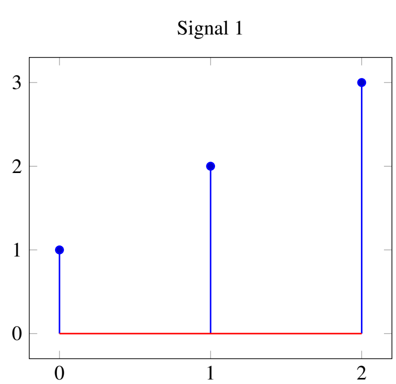
(b) Linearity

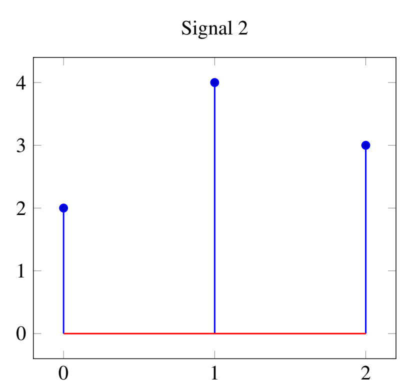
**Answer:** Let $\alpha, \beta, w_i, x_i, z_i \in \mathbb{R}$.

$$\begin{aligned}\left\langle \alpha\begin{bmatrix}v_1\\v_2\end{bmatrix} + \beta\begin{bmatrix}w_1\\w_2\end{bmatrix}, \begin{bmatrix}z_1\\z_2\end{bmatrix}\right\rangle &= \left\langle \begin{bmatrix}\alpha v_1 + \beta w_1\\ \alpha v_2 + \beta w_2\end{bmatrix}, \begin{bmatrix}z_1\\z_2\end{bmatrix}\right\rangle\\ &= (\alpha v_1 + \beta w_1)z_1 + (\alpha v_2 + \beta w_2)z_2\\ &= \alpha(v_1 z_1 + v_2 z_2) + \beta(w_1 z_1 + w_2 z_2)\\ &= \alpha v_1 z_1 + \alpha v_2 z_2 + \beta w_1 z_1 + \beta w_2 z_2\\ &= \alpha\left\langle \begin{bmatrix}v_1\\v_2\end{bmatrix}, \begin{bmatrix}z_1\\z_2\end{bmatrix}\right\rangle + \beta\left\langle \begin{bmatrix}w_1\\w_2\end{bmatrix}, \begin{bmatrix}z_1\\z_2\end{bmatrix}\right\rangle\end{aligned}$$

**3. Correlation**

We are given the following two signals, $s_1[n]$ and $s_2[n]$ respectively.



Find the cross correlations, $\text{corr}_{s_1}(s_2)$ and $\text{corr}_{s_2}(s_1)$ for signals $s_[n]$ and $s_2[n]$. Recall

$$\text{corr}_x(y)[k] = \sum_{i=-\infty}^{\infty} x[i]y[i-k].$$

$\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$

| $\vec{s}_1$ | 0 | 0 | 1 | 2 | 3 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n+2]$ | | | | | | | |
| $\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$ | + | + | + | + | + | + | = |

| $\vec{s}_1$ | 0 | 0 | 1 | 2 | 3 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n+1]$ | | | | | | | |
| $\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$ | + | + | + | + | + | + | = |

| $\vec{s}_1$ | 0 | 0 | 1 | 2 | 3 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n]$ | | | | | | | |
| $\langle \vec{s}_1, \vec{s}_2[n] \rangle$ | + | + | + | + | + | + | = |

| $\vec{s}_1$ | 0 | 0 | 1 | 2 | 3 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n-1]$ | | | | | | | |
| $\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$ | + | + | + | + | + | + | = |

| $\vec{s}_1$ | 0 | 0 | 1 | 2 | 3 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n-2]$ | | | | | | | |
| $\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$ | + | + | + | + | + | + | = |

$\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$

| $\vec{s}_2$ | 0 | 0 | 2 | 4 | 3 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\vec{s}_1[n+2]$ | | | | | | | |
| $\langle \vec{s}_2, \vec{s}_1[n+2] \rangle$ | + | + | + | + | + | + | = |

| $\vec{s}_2$ | 0 | 0 | 2 | 4 | 3 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\vec{s}_1[n+1]$ | | | | | | | |
| $\langle \vec{s}_2, \vec{s}_1[n+1] \rangle$ | + | + | + | + | + | + | = |

| $\vec{s}_2$ | 0 | 0 | 2 | 4 | 3 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\vec{s}_1[n]$ | | | | | | | |
| $\langle \vec{s}_2, \vec{s}_1[n] \rangle$ | + | + | + | + | + | + | = |

| $\vec{s}_2$ | 0 | | 0 | | 2 | | 4 | | 3 | | 0 | | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{s}_1[n-1]$ | | | | | | | | | | | | | | |
| $\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$ | | + | | + | | + | | + | | + | | + | | = |

| $\vec{s}_2$ | 0 | | 0 | | 2 | | 4 | | 3 | | 0 | | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{s}_1[n-2]$ | | | | | | | | | | | | | | |
| $\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$ | | + | | + | | + | | + | | + | | + | | = |

**Answer:** The linear cross-correlation is calculated by shifting the second signal both forward and backward until there is no overlap between the signals. When there is no overlap, the cross-correlation goes to zero. Both of these cross-correlations should have only zeros outside the range: $-2 \leq n \leq 2$.

$\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$

| $\vec{s}_1$ | 0 | | 0 | | 1 | | 2 | | 3 | | 0 | | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n+2]$ | 2 | | 4 | | 3 | | 0 | | 0 | | 0 | | 0 | |
| $\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$ | 0 | + | 0 | + | 3 | + | 0 | + | 0 | + | 0 | + | 0 | = 3 |

| $\vec{s}_1$ | 0 | | 0 | | 1 | | 2 | | 3 | | 0 | | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n+1]$ | 0 | | 2 | | 4 | | 3 | | 0 | | 0 | | 0 | |
| $\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$ | 0 | + | 0 | + | 4 | + | 6 | + | 0 | + | 0 | + | 0 | = 10 |

| $\vec{s}_1$ | 0 | | 0 | | 1 | | 2 | | 3 | | 0 | | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n]$ | 0 | | 0 | | 2 | | 4 | | 3 | | 0 | | 0 | |
| $\langle \vec{s}_1, \vec{s}_2[n] \rangle$ | 0 | + | 0 | + | 2 | + | 8 | + | 9 | + | 0 | + | 0 | = 19 |

| $\vec{s}_1$ | 0 | | 0 | | 1 | | 2 | | 3 | | 0 | | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n-1]$ | 0 | | 0 | | 0 | | 2 | | 4 | | 3 | | 0 | |
| $\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$ | 0 | + | 0 | + | 0 | + | 4 | + | 12 | + | 0 | + | 0 | = 16 |

| $\vec{s}_1$ | 0 | | 0 | | 1 | | 2 | | 3 | | 0 | | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n-2]$ | 0 | | 0 | | 0 | | 0 | | 2 | | 4 | | 3 | |
| $\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$ | 0 | + | 0 | + | 0 | + | 0 | + | 6 | + | 0 | + | 0 | = 6 |

$\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$

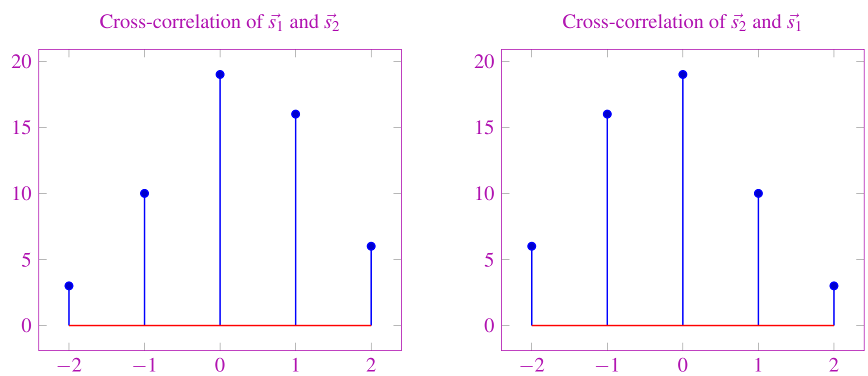
| $\vec{s}_2[n]$ | 0 | | 0 | | 2 | | 4 | | 3 | | 0 | | 0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{s}_1[n+2]$ | 1 | | 2 | | 3 | | 0 | | 0 | | 0 | | 0 | |
| $\langle \vec{s}_2, \vec{s}_1[n+2] \rangle$ | 0 | + | 0 | + | 6 | + | 0 | + | 0 | + | 0 | + | 0 | = 6 |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n]$ | 0 | | 0 | | 2 | | 4 | | 3 | | 0 | | 0 | |
| $\vec{s}_1[n+1]$ | 0 | | 1 | | 2 | | 3 | | 0 | | 0 | | 0 | |
| $\langle \vec{s}_2, \vec{s}_1[n+1] \rangle$ | 0 | + | 0 | + | 4 | + | 12 | + | 0 | + | 0 | + | 0 | $= 16$ |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n]$ | 0 | | 0 | | 2 | | 4 | | 3 | | 0 | | 0 | |
| $\vec{s}_1[n]$ | 0 | | 0 | | 1 | | 2 | | 3 | | 0 | | 0 | |
| $\langle \vec{s}_2, \vec{s}_1[n] \rangle$ | 0 | + | 0 | + | 2 | + | 8 | + | 9 | + | 0 | + | 0 | $= 19$ |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n]$ | 0 | | 0 | | 2 | | 4 | | 3 | | 0 | | 0 | |
| $\vec{s}_2[n-1]$ | 0 | | 0 | | 0 | | 1 | | 2 | | 3 | | 0 | |
| $\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$ | 0 | + | 0 | + | 0 | + | 4 | + | 6 | + | 0 | + | 0 | $= 10$ |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{s}_2[n]$ | 0 | | 0 | | 2 | | 4 | | 3 | | 0 | | 0 | |
| $\vec{s}_2[n-2]$ | 0 | | 0 | | 0 | | 0 | | 1 | | 2 | | 3 | |
| $\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$ | 0 | + | 0 | + | 0 | + | 0 | + | 3 | + | 0 | + | 0 | $= 3$ |




Notice that $\text{corr}_{\vec{s}_1}(\vec{s}_2)[k] = \text{corr}_{\vec{s}_2}(\vec{s}_1)[-k]$, i.e. changing the order of the signals reverses the cross-correlation sequence.