EECS 16A Designing Information Devices and Systems I
Summer 2020 Discussion 6B

Reference: Inner products

Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be vectors in real vector space $\mathbb{V}$. A mapping $\langle \cdot, \cdot \rangle$ is said to be an inner product on $\mathbb{V}$ if it satisfies the following three properties:

(a) Symmetry: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

(b) Linearity: $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$ and $\langle c\vec{x}, \vec{y} \rangle = c \langle \vec{x}, \vec{y} \rangle$

(c) Positive-definiteness: $\langle \vec{x}, \vec{x} \rangle \geq 0$, with equality if and only if $\vec{x} = \vec{0}$.

We define the norm of $\vec{x}$ as $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$.

Cross-correlation:

The cross-correlation between two signals $r[n]$ and $s[n]$ is defined as follows:

$$\text{corr}_r(s)[k] = \sum_{i=-\infty}^{\infty} r[i]s[i-k].$$

1. Geometric Interpretation of the Inner Product In this problem, we will explore the geometric interpretation of the Euclidean inner product, restricting ourselves to vectors in $\mathbb{R}^2$.

(a) For each of the following cases, pick two vectors that satisfy the condition and find the inner product.

i. Parallel Vectors

Answer: Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\langle \vec{x}, \vec{x} \rangle = 1 \cdot 1 + 1 \cdot 1 = 2$$

ii. Anti-parallel

Answer: Again, let $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\langle \vec{x}, -\vec{x} \rangle = 1 \cdot (-1) + 1 \cdot (-1) = -2 = -\langle \vec{x}, \vec{x} \rangle$$

iii. Perpendicular

Answer: Let $\vec{x} = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{y} = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot 0 + 0 \cdot 1 = 0$$

(b) Now, derive a formula for the inner product of two vectors in terms of their magnitudes and the angle between them.

Answer: From trigonometric calculation, if $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then we know that $x_1 = \|\vec{x}\| \cdot \cos\alpha$, $x_2 = \|\vec{x}\| \cdot \sin\alpha$, $y_1 = \|\vec{y}\| \cdot \cos\beta$ and $y_2 = \|\vec{y}\| \cdot \sin\beta$ (as in the figure). Then you can directly write

$$\begin{aligned}
\langle \vec{x}, \vec{y} \rangle &= x_1 \cdot y_1 + x_2 \cdot y_2 = \\
&= \underbrace{\|\vec{x}\| \cdot \cos\alpha}_{x_1} \cdot \underbrace{\|\vec{y}\| \cdot \cos\beta}_{y_1} + \underbrace{\|\vec{x}\| \cdot \sin\alpha}_{x_2} \cdot \underbrace{\|\vec{y}\| \cdot \sin\beta}_{y_2} \\
&= \|\vec{x}\| \|\vec{y}\| \left(\cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta\right) = \\
&= \|\vec{x}\| \|\vec{y}\| \cdot \cos\left(\beta - \alpha\right) \\
&= \|\vec{x}\| \|\vec{y}\| \cdot \cos\theta
\end{aligned}$$

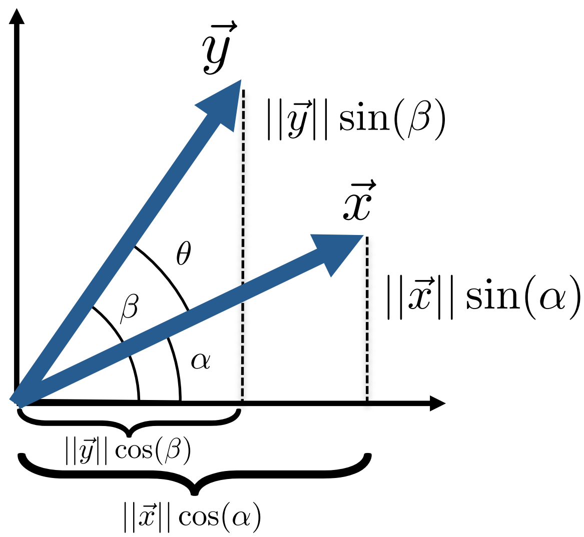
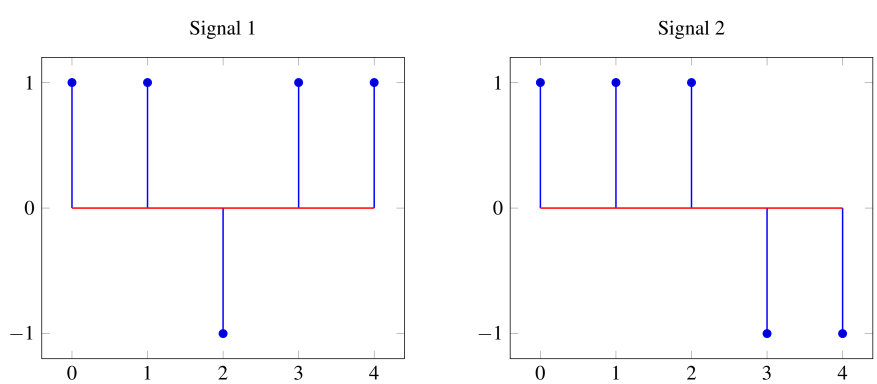


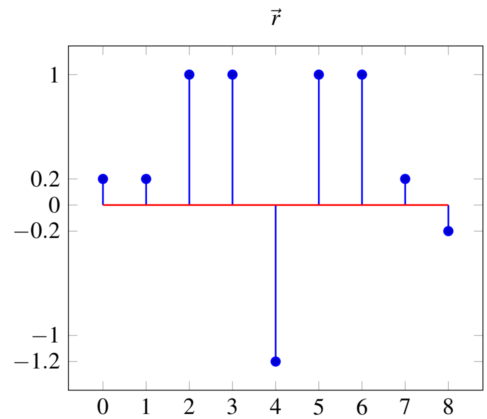
Figure 1: Two general vectors in $\mathbb{R}^2$

2. Identifying satelites and their delays

We are given the following two signals, $\vec{s}_1$ and $\vec{s}_2$ respectively, that are signatures for two satellites.



(a) Your cellphone antenna receives the following signal $r[n]$. You know that there may be some noise present in $r[n]$ in addition to the transmission from the satellite.



Which satellites are transmitting? What is the delay between the satellite and your cellphone? Use cross-correlation to justify your answer. You can use iPython to compute the cross-correlation.

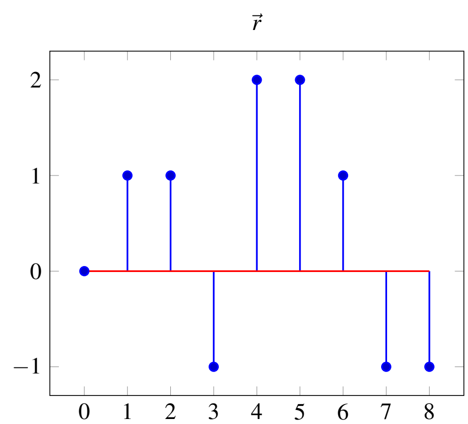
Answer: We calculate both $\text{corr}_r(\vec{s}_1)[k]$ and $\text{corr}_r(\vec{s}_2)[k]$:

$$\begin{aligned}
&\text{corr}_{\vec{r}}(\vec{s}_1)[-4] = (0.2)(1) = 0.2\\
&\text{corr}_{\vec{r}}(\vec{s}_1)[-3] = (0.2)(1) + (0.2)(1) = 0.4\\
&\text{corr}_{\vec{r}}(\vec{s}_1)[-2] = (1)(1) + (0.2)(1) + (0.2)(-1) = 1\\
&\text{corr}_{\vec{r}}(\vec{s}_1)[-1] = (1)(1) + (1)(1) + (0.2)(-1) + (0.2)(1) = 2\\
&\text{corr}_{\vec{r}}(\vec{s}_1)[0] = (-1.2)(1) + (1)(1) + (1)(-1) + (0.2)(1) + (0.2)(1) = -0.8\\
&\text{corr}_{\vec{r}}(\vec{s}_1)[1] = (1)(1) + (-1.2)(1) + (1)(-1) + (1)(1) + (0.2)(1) = 0\\
&\text{corr}_{\vec{r}}(\vec{s}_1)[2] = (1)(1) + (1)(1) + (-1.2)(-1) + (1)(1) + (1)(1) = \mathbf{5.2}\\
&\text{corr}_{\vec{r}}(\vec{s}_1)[3] = (0.2)(1) + (1)(1) + (1)(-1) + (-1.2)(1) + (1)(1) = 0\\
&\text{corr}_{\vec{r}}(\vec{s}_1)[4] = (-0.2)(1) + (0.2)(1) + (1)(-1) + (1)(1) + (-1.2)(1) = -1.2\\
&\text{corr}_{\vec{r}}(\vec{s}_1)[5] = (-0.2)(1) + (0.2)(-1) + (1)(1) + (1)(1) = 1.6\\
&\text{corr}_{\vec{r}}(\vec{s}_1)[6] = (-0.2)(-1) + (0.2)(1) + (1)(1) = 1.4\\
&\text{corr}_{\vec{r}}(\vec{s}_1)[7] = (-0.2)(1) + (0.2)(1) = 0\\
&\text{corr}_{\vec{r}}(\vec{s}_1)[8] = (-0.2)(1) = -0.2
\end{aligned}$$

$$\text{corr}_{\vec{r}}(\vec{s}_2)[-4] = (0.2)(-1) = -0.2$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[-3] = (0.2)(-1) + (0.2)(-1) = -0.4$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[-2] = (1)(-1) + (0.2)(-1) + (0.2)(1) = -1$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[-1] = (1)(-1) + (1)(-1) + (0.2)(1) + (0.2)(1) = -1.6$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[0] = (-1.2)(-1) + (1)(-1) + (1)(1) + (0.2)(1) + (0.2)(1) = 1.6$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[1] = (1)(-1) + (-1.2)(-1) + (1)(1) + (1)(1) + (0.2)(1) = 2.4$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[2] = (1)(-1) + (1)(-1) + (-1.2)(1) + (1)(1) + (1)(1) = -1.2$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[3] = (0.2)(-1) + (1)(-1) + (1)(1) + (-1.2)(1) + (1)(1) = -0.4$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[4] = (-0.2)(-1) + (0.2)(-1) + (1)(1) + (1)(1) + (-1.2)(1) = 0.8$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[5] = (-0.2)(-1) + (0.2)(1) + (1)(1) + (1)(1) = 2.4$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[6] = (-0.2)(1) + (0.2)(1) + (1)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[7] = (-0.2)(1) + (0.2)(1) = 0$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[8] = (-0.2)(1) = -0.2$$

The maximum correlation value is 5.2 at $k = 2$. Since we have a plus-minus 1 signal of length 5, this high correlation likely comes from the satellite 1 transmission.

(b) Now your cellphone receives a new signal $r[n]$ as below. What the satellites that are transmitting and what is the delay between each satellite and your cellphone?



Answer: We want to find shifts k_1 and k_2 such that: $\vec{r}[n] = \vec{s}_1[n-k_1] + \vec{s}_2[n-k_2]$.
We calculate both $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$ and $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$ for different shifts k. The index where the maximum correlation value is achieved will tell us the shift indices (delays).

$$\text{corr}_{\vec{r}}(\vec{s}_1)[-3] = (1)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[-2] = (1)(1) + (1)(1) = 2$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[-1] = (-1)(1) + (1)(1) + (1)(-1) = -1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[0] = (2)(1) + (-1)(1) + (1)(-1) + (1)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[1] = (2)(1) + (2)(1) + (-1)(-1) + (1)(1) + (1)(1) = \mathbf{7}$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[2] = (1)(1) + (2)(1) + (2)(-1) + (-1)(1) + (1)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[3] = (-1)(1) + (1)(1) + (2)(-1) + (2)(1) + (-1)(1) = -1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[4] = (-1)(1) + (-1)(1) + (1)(-1) + (2)(1) + (2)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[5] = (-1)(1) + (-1)(-1) + (1)(1) + (2)(1) = 3$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[6] = (-1)(-1) + (-1)(1) + (1)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[7] = (-1)(1) + (-1)(1) = -2$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[8] = (-1)(1) = -1$$

$$\text{corr}_{\vec{r}}(\vec{s}_2)[-3] = (1)(-1) = -1$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[-2] = (1)(-1) + (1)(-1) = -2$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[-1] = (-1)(-1) + (1)(-1) + (1)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[0] = (2)(-1) + (-1)(-1) + (1)(1) + (1)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[1] = (2)(-1) + (2)(-1) + (-1)(1) + (1)(1) + (1)(1) = -3$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[2] = (1)(-1) + (2)(-1) + (2)(1) + (-1)(1) + (1)(1) = -1$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[3] = (-1)(-1) + (1)(-1) + (2)(1) + (2)(1) + (-1)(1) = 3$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[4] = (-1)(-1) + (-1)(-1) + (1)(1) + (2)(1) + (2)(1) = \mathbf{7}$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[5] = (-1)(-1) + (-1)(1) + (1)(1) + (2)(1) = 3$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[6] = (-1)(1) + (-1)(1) + (1)(1) = -1$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[7] = (-1)(1) + (-1)(1) = -2$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[8] = (-1)(1) = -1$$

The maximum correlation between signals $\vec{r}$ and $\vec{s}_1$ was achieved at $k_1 = 1$, and the maximum correlation between signals $\vec{r}$ and $\vec{s}_2$ was achieved at $k_2 = 4$.