EECS 16A Designing Information Devices and Systems I
Summer 2020 Discussion 2D

1. Aragorn's Odyssey

In a desperate attempt to save Minas Tirith, Aragorn is trying to maneuver your ship in a 2D plane around the fleet of the Corsairs of the South. The position of your ship in two dimensions (x, y) is represented as a vector, $\begin{bmatrix} x \\ y \end{bmatrix}$.

(a) In order to evade the Witch-King of Angmar, Gandalf provides Aragorn with linear transformation spell. The spell first reflects your ship along the X-axis (i.e. multiplies the Y-coordinate by -1) and then rotates it by 30 degrees counterclockwise. Express the transformation Gandalf's spell performed on the ship's location as a 2×2 matrix.

Hint: Recall that the matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *rotates a vector counterclockwise by* θ.

(b) If the ship was initially 1 unit distance away from the origin $(0, 0)$, how far is it from the origin after the transformation above? Justify your answer.

(c) Having evaded the Witch-King and the Corsairs, Aragorn needs to quickly reach Minas Tirith. To do so, he uses the wind spell, $\mathbf{B}_{\text{spell}}$, ten times, where his position $\vec{x}[t]$ changes according to the equation

$$\vec{x}[t+1] = \mathbf{B}_{\text{spell}}\vec{x}[t],$$

where $\mathbf{B}_{\text{spell}} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$.

The initial location of your ship is $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. **What is the location of your ship at time** $t = 10$**, i.e. what is** $\vec{x}[10]$**?** Explicitly compute your final solution and justify your answer.

(d) The ship is now moving in an n dimensional space. The position of the ship at time t is represented by $\vec{x}[t] \in \mathbb{R}^n$. The ship starts at the origin $\vec{0}$.

Aragorn tries a new spell, $\mathbf{C}_{\text{spell}} \in \mathbb{R}^{n\times n}$, $\mathbf{C}_{\text{spell}} \neq 0$. In addition to the spell, the ship is given some ability to steer using the scalar input $u[t] \in \mathbb{R}$. The location of the ship at the next time step is described by the equation:

$$\vec{x}[t+1] = \mathbf{C}_{\text{spell}}\vec{x}[t] + \vec{b}u[t],$$

where $\vec{b} \in \mathbb{R}^n$ is fixed.

You know from the Segway problem on the homework that the ship can reach all locations in the $\text{span}\{\vec{b}, \mathbf{C}_{\text{spell}}\vec{b}, \mathbf{C}_{\text{spell}}^2\vec{b}, \cdots, \mathbf{C}_{\text{spell}}^9\vec{b}\}$ in ten time steps. **Suppose we tell you that** $\vec{b} \neq 0$ **is an eigenvector of** $\mathbf{C}_{\text{spell}}$. What is the maximum dimension of the subspace of locations the ship can reach? Justify your answer.

2. Trouble in Telecomm

Fred (x_0), Tina (x_1), and Will (x_2) each are sending messages (where each message x_0, x_1, x_2 is a real number) at the same time to Alec, Kristin, and Colin respectively.

To achieve this, the phone company will transmit $\vec{y}$, which is a vector of linear combinations of x_0, x_1, x_2. Specifically,

$$\vec{y} = \mathbf{V}\vec{x} = \begin{bmatrix} | & | & | \\ \vec{c}_0 & \vec{c}_1 & \vec{c}_2 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}. \tag{1}$$

V is the encoding matrix.

On the receiver side, Alec, Kristin and Colin need to recover x_0, x_1, x_2 respectively from $\vec{y}$. You are helping the phone company evaluate different choices for the columns $\vec{c}_0$, $\vec{c}_1$ and $\vec{c}_2$ of matrix **V**:

$$\begin{aligned} \mathbf{V_0} &= \begin{bmatrix} | & | & | \\ \vec{c}_0 & \vec{c}_1 & \vec{c}_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 10 \\ 0 & 2 & 4 \end{bmatrix} \\ \mathbf{V_1} &= \begin{bmatrix} | & | & | \\ \vec{c}_0 & \vec{c}_1 & \vec{c}_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{aligned} \tag{2}$$

(a) You decide to characterize $\mathbf{V_0}$ in terms of its null space. Find a basis for the nullspace of $\mathbf{V_0}$.

(b) If the matrix $\mathbf{V_0} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 10 \\ 0 & 2 & 4 \end{bmatrix}$ is invertible, find its inverse. If it is not invertible, why not? Given this, is $\mathbf{V_0}$ a good encoding matrix to use? Justify your answer.

(c) If the matrix $\mathbf{V_1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is invertible, find its inverse. If it is not invertible, why not? Given this, is $\mathbf{V_1}$ a good encoding matrix to use? Justify your answer.

3. Free-form review with discussion section TAs (if time)