①

$$\det(A - \lambda I) = 0$$

1a.

$$M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \qquad \lambda_1 = 1,\quad \lambda_2 = 2,\quad \lambda_3 = \frac{1}{2}$$

$\lambda_1 = 1$

$$[M - \lambda I \mid \vec{0}\,] = \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 + \frac{1}{2}R_2 \rightarrow \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-\frac{1}{2}R_2 \rightarrow \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = 0 \qquad -\frac{1}{2}x_1 + \frac{1}{2}x_2 = 0 \qquad x_1 = x_2$

$$\vec{v}_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \alpha \in \mathbb{R}$$

②

$\lambda_2 = 2$

$$[M - \lambda I \mid \vec{0}\,] = \left[\begin{array}{ccc|c} -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$2R_1 + R_2 \rightarrow R_1 \quad \left[\begin{array}{ccc|c} -3 & 0 & -3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -x_3$

$x_2 = -2x_3$

$$\rightarrow \vec{v}_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \beta, \quad \beta \in \mathbb{R}$$

In chat/verbally stated

$$\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \gamma, \quad \gamma \in \mathbb{R}$$

③ $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$

$\lambda_1 = 1$

$\lambda_2 = 2$

$\lambda_3 = 1/2$

$\lim_{n\to\infty} M^n\vec{x}$

$$M^n\vec{x} = \alpha(1)^n\vec{v}_1 + \beta(2)^n\vec{v}_2 + \gamma(1/2)^n\vec{v}_3$$

| $\alpha$ | $\beta$ | $\gamma$ | converge? | $\lim_{n\to\infty} M^n\vec{x}$ |
|---|---|---|---|---|
| 0 | 0 | $\neq 0$ | Y | 0 |
| 0 | $\neq 0$ | 0 | N | X |
| 0 | $\neq 0$ | $\neq 0$ | N | X |
| $\neq 0$ | 0 | 0 | Y | $\alpha\vec{v}_1$ |
| $\neq 0$ | 0 | $\neq 0$ | Y | $\alpha\vec{v}_1$ |
| $\neq 0$ | $\neq 0$ | 0 | N | X |
| $\neq 0$ | $\neq 0$ | $\neq 0$ | N | X |

④ $P_2$, $p(t) = at^2 + bt + c$

$\{t^2, t, 1\}$

$T_1(f(t)) = 2f(t)$

$T_2(f(t)) = f'(t)$

$$T_1: \quad A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$f(t)$ has coordinates $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$f(t) \sim \vec{x}$

$2f(t) \sim A_1\vec{x}$

$$A_2\vec{x} = \begin{bmatrix} 0 \\ 2a \\ b \end{bmatrix}$$

$$T_2: \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$f(t) \sim \vec{x}$ $\qquad at^2 + bt + c$

$f'(t) \sim A_2\vec{x}$ $\qquad 2at + b$