EECS16A DIS 1A

Concepts from lecture that will come up

Span, proofs, linear combinations, consistency of a system of linear equations/ consistent system of linear equations

Skills/Things to take away

① 3 ways of reasoning about the span of a set of vectors

- ↳ Definition
- ↳ Geometrically
- ↳ Checking consistency

② Proof: How to show equality of sets

Definitions

① Span (of a set of vectors): the set of all linear combinations of a set of vectors

Example

$$\underbrace{\text{span}\left\{\underbrace{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}2\\2\\1\end{bmatrix},\begin{bmatrix}3\\1\\1\end{bmatrix}}_{\text{set of vectors}}\right\} = \left\{\alpha_0\begin{bmatrix}1\\2\\0\end{bmatrix}+\beta_0\begin{bmatrix}2\\2\\1\end{bmatrix}+\gamma_0\begin{bmatrix}3\\1\\1\end{bmatrix},\ \alpha_1\begin{bmatrix}1\\2\\0\end{bmatrix}+\beta_1\begin{bmatrix}2\\2\\1\end{bmatrix}+\gamma_1\begin{bmatrix}3\\1\\1\end{bmatrix}, \ldots \text{infinitely more vectors}\right\}}_{\text{larger set of vectors generated by taking all linear combinations}}$$

$\boxed{1}$ (a) What is $\text{span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$?

↳ What does a single vector in $\text{span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$ look like (as an expression)?

$$\alpha_1 \begin{bmatrix}1\\2\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}1\\2\\0\end{bmatrix}$$

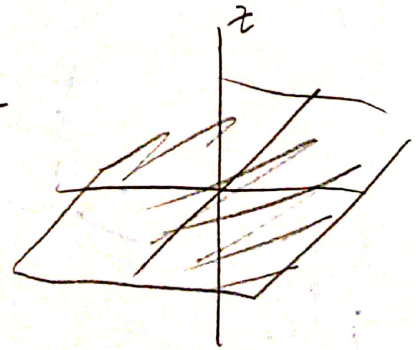
↳ So the entire set is:

$$\text{Span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\} = \left\{ \alpha_1 \begin{bmatrix}1\\2\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\1\\0\end{bmatrix} \;\middle|\; \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

↳ Can use inspection to write it more simply (not always immediately possible)

e.g. higher dimensional spaces

$$\begin{bmatrix}s\\t\\0\end{bmatrix} \quad s, t \in \mathbb{R}$$



1 ⓑ Is $\begin{bmatrix}5\\5\\0\end{bmatrix}$ in span $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$

$$\Rightarrow \begin{bmatrix}5\\5\\0\end{bmatrix} = \alpha_1\begin{bmatrix}1\\2\\0\end{bmatrix} + \alpha_2\begin{bmatrix}2\\1\\0\end{bmatrix}$$

If α_1 and α_2 exist that make the above eqn. true, $\begin{bmatrix}5\\5\\0\end{bmatrix}$ is in the span $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$

$$\underbrace{\begin{bmatrix}1 & 2\\2 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} = \begin{bmatrix}5\\5\\0\end{bmatrix}}_{A\vec{x}=\vec{b}} \rightarrow \left[\begin{array}{cc|c}1 & 2 & 5\\2 & 1 & 5\\0 & 0 & 0\end{array}\right]$$

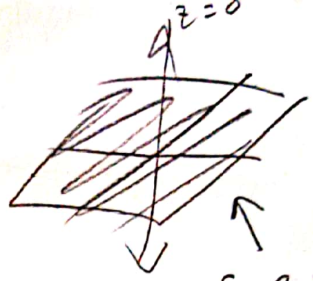
$$\frac{5}{3}\begin{bmatrix}1\\2\\0\end{bmatrix} + \frac{5}{3}\begin{bmatrix}2\\1\\0\end{bmatrix} \stackrel{?}{=} \checkmark \begin{bmatrix}5\\5\\0\end{bmatrix}$$

$[0 \; 0 \; 0 \;|\; 1 \; 1]$

if inconsistent $\Rightarrow$ vector is not in the span

Checking if a vector is in a span of vectors $\Leftrightarrow$ a solution existing to a system of equations

ⓒ What $\vec{v}$ makes $\text{span}\left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \vec{v} \right\} = \mathbb{R}^3$?



$\begin{bmatrix}s\\t\\0\end{bmatrix}$ inside

$\text{span}\left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix} \right\}$

$$\vec{v} = \begin{bmatrix}0\\0\\1\end{bmatrix} \rightarrow \text{span}\left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\} = \mathbb{R}^3$$

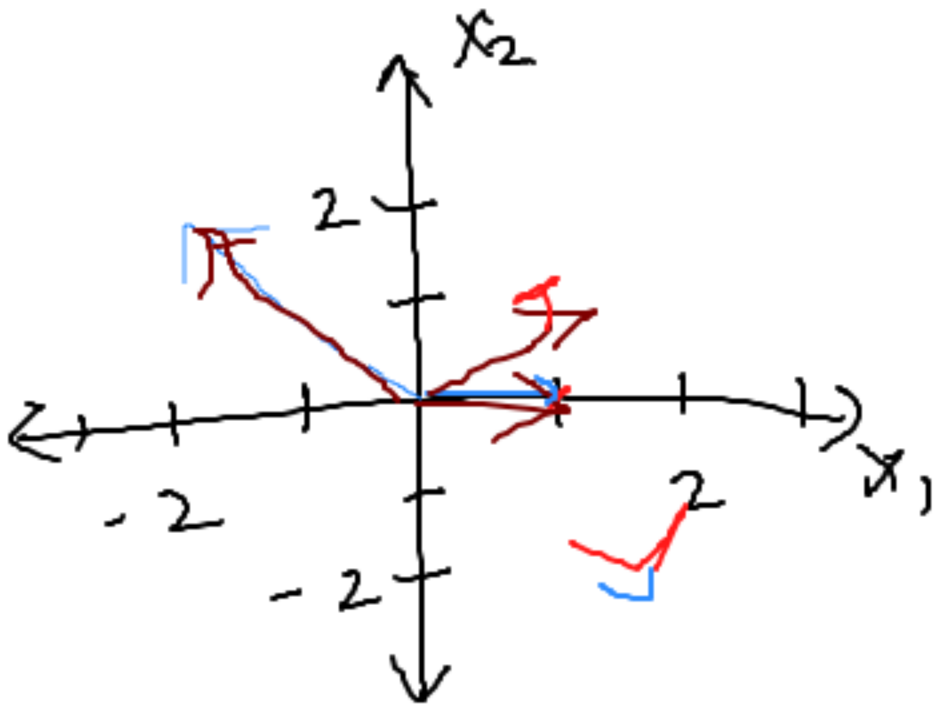
$$\vec{v} = \begin{bmatrix}3\\3\\-10\end{bmatrix} \Rightarrow \text{span}\left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\3\\-10\end{bmatrix} \right\} \overset{?}{=} \mathbb{R}^3 \checkmark$$

ⓓ

$$\begin{bmatrix}1 & 2\\2 & 1\\0 & 0\end{bmatrix} \vec{x} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\hookrightarrow \left[\begin{array}{cc|c}1 & 2 & b_1\\2 & 1 & b_2\\0 & 0 & b_3\end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|c}1 & 2 & b_1\\0 & -3 & b_2-2b_1\\0 & 0 & b_3\end{array}\right] \xrightarrow{R_2/-3} \left[\begin{array}{cc|c}1 & 2 & b_1\\0 & 1 & \frac{2b_1-b_2}{3}\\0 & 0 & b_3\end{array}\right]$$

$$\xrightarrow{R_1-2R_2} \left[\begin{array}{cc|c}1 & 0 & \frac{2b_2-b_1}{3}\\0 & 1 & \frac{2b_1-b_2}{3}\\0 & 0 & b_3\end{array}\right] \Rightarrow b_3 = 0 \text{ for system to be consistent}$$

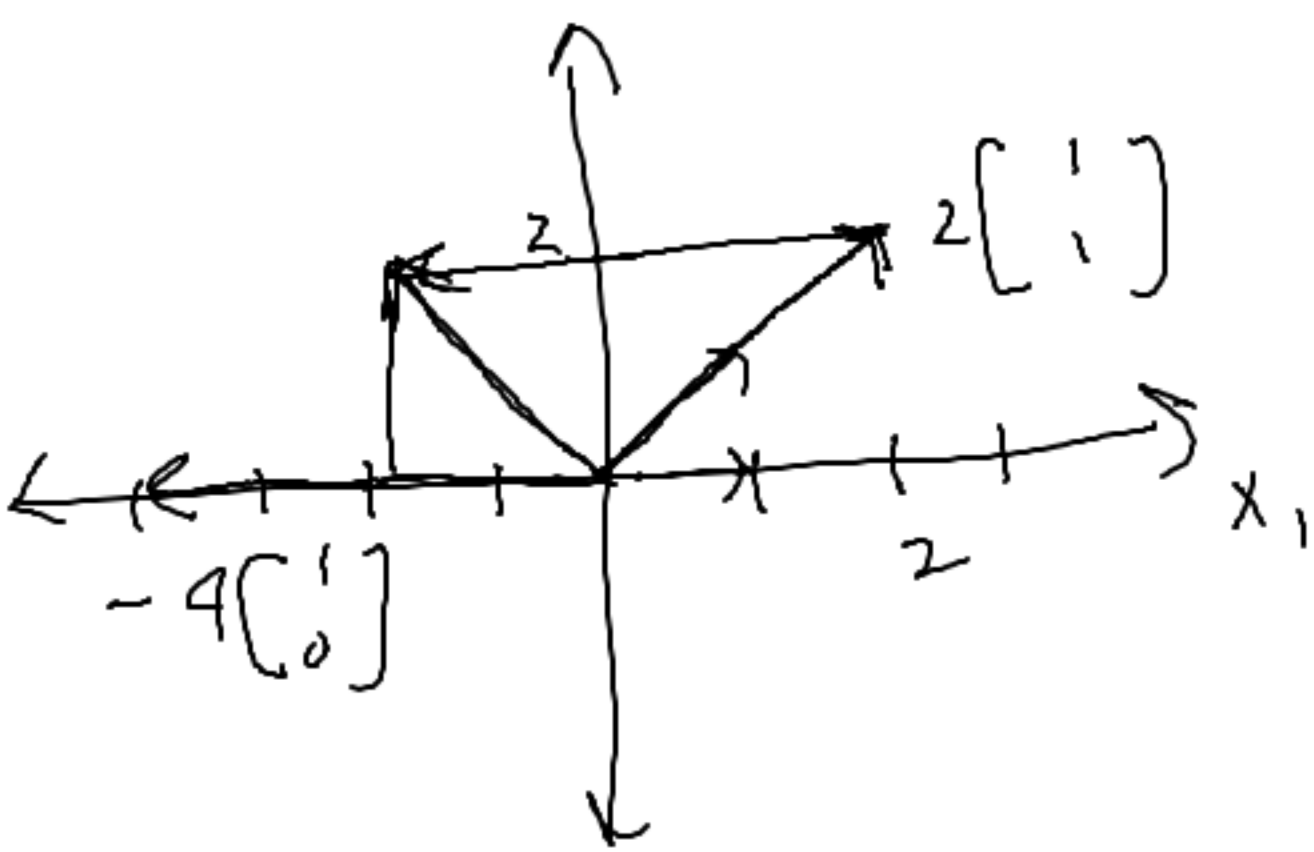


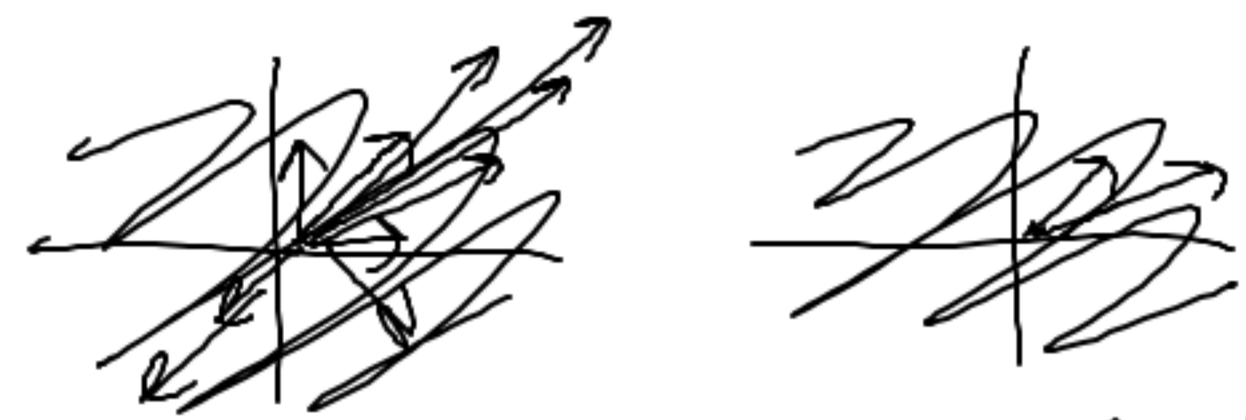
$\alpha, \beta \quad \alpha\begin{bmatrix}1\\0\end{bmatrix}+\beta\begin{bmatrix}1\\1\end{bmatrix}=\overset{\vec{c}}{\begin{bmatrix}-2\\2\end{bmatrix}}$

$\alpha=-4 \quad \beta=2$

$\vec{a}=\begin{bmatrix}1\\0\end{bmatrix} \quad \vec{b}=\begin{bmatrix}1\\1\end{bmatrix} \quad \vec{c}=\begin{bmatrix}-2\\2\end{bmatrix}$

$\vec{c} \in \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$





$\begin{bmatrix}-2\\2\end{bmatrix} \in \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}\right\}$

$\begin{bmatrix}1&2\\1&1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}=\begin{bmatrix}-2\\2\end{bmatrix}$

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

known

$$A \vec{x} = \vec{b}$$

unknown

$$\vec{c} = \alpha \vec{a} + \beta \vec{b}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underline{\alpha} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \underline{\beta} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Sets : how to check if equal

$\{ A, B, C \}$ $\{ A, B, C, D \}$

Two sets are equal if every element in one set appears in the other and vice versa.

$$\underbrace{\text{span}\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}}_{\text{is inside}} \stackrel{?}{=} \text{span}\{ \alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$$

$\alpha \neq 0$

are vectors here also in the first set?

$$\vec{u} = \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \cdots + \beta_n \vec{v}_n$$

$$\vec{u} \stackrel{?}{=} \gamma_1 (\alpha \vec{v}_1) + \gamma_2 \vec{v}_2 + \cdots + \gamma_n \vec{v}_n$$

$$\beta_1 = \gamma_1 \alpha \qquad \gamma_2 = \beta_2, \ \gamma_3 = \beta_3, \ \dots \ \gamma_n = \beta_n$$

$\vec{w} \in \text{Span}\{\alpha\vec{v}_1, \vec{v}_2, \dots \vec{v}_n$

$\vec{w} = p_1(\alpha\vec{v}_1) + p_2\vec{v}_2 + \dots + p_n\vec{v}_n$

If we want $\vec{w}$ to be in the first set,

$\vec{w} = q_1\vec{v}_1 + q_2\vec{v}_2 + \dots + q_n\vec{v}_n$

$p_2 = q_2, \dots, p_n = q_n$

$p_1\alpha = q_1$

$p_1 = \frac{q_1}{\alpha}$

By choosing these values of $\vec{p}$,
$\vec{w}$ belongs to the first set.
$\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = \text{Span}\{\alpha\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$