EECS16A DIS 2A

1. How to check if a set of vectors is a subspace
2. How to find columnspace and nullspace of a matrix A
3. Calculating 2x2 determinants

How to tell if a set of vectors is a subspace?

$S = \{ \vec{v}_1, \vec{v}_2, \dots$ infinitely many $\}$

[all vectors in S must be in V
S is a subset of V]

For S to be a subspace of a vectorspace V,

1. for any vectors $\vec{v}_1, \vec{v}_2$ in S
   $\vec{v}_1 + \vec{v}_2$ must also be in S (closure under vector addition)

2. for any vector $\vec{v}$ in S and any scalar α in $\mathbb{R}$
   $\alpha \vec{v}$ is also in S

② $\Rightarrow$ ③ $\vec{0}$ must be in S.

② is true for $\alpha = 0$

$0 \cdot \vec{v} \Rightarrow$ is in S

helps with checking that some set is <u>not</u> a subspace quickly

Is $V = \left\{ \vec{v} \mid \vec{v} = c\begin{bmatrix}1\\1\\1\end{bmatrix} + d\begin{bmatrix}1\\0\\1\end{bmatrix}, c, d \in \mathbb{R} \right\}$

set of all $\vec{v}$

such that

① Write two general vectors belonging to the set

② Check that the sum is also in that set (closure under vec. add.)

$$\vec{v}_1 = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + d_1\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\vec{v}_2 = c_2\begin{bmatrix}1\\1\\1\end{bmatrix} + d_2\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Check that $\vec{v}_1 + \vec{v}_2$ takes the same form as vectors in V (our set)

$$\vec{v}_1 + \vec{v}_2 = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + d_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\1\end{bmatrix} + d_2\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$= (c_1+c_2)\begin{bmatrix}1\\1\\1\end{bmatrix} + (d_1+d_2)\begin{bmatrix}1\\0\\1\end{bmatrix}$$

, $\vec{v}_1 + \vec{v}_2$ is in V

V is closed under vector addition

(2) Write a general expression for a vector $(\vec{v})$ and a scalar (α)

$\Rightarrow$ Check that $\alpha\vec{v}$ is in the set (the same form of vector)

$$\vec{v} = c\begin{bmatrix}1\\1\\1\end{bmatrix} + d\begin{bmatrix}1\\0\\1\end{bmatrix} \qquad \alpha \in \mathbb{R}$$

$$\alpha\vec{v} = \underline{\alpha c}\begin{bmatrix}1\\1\\1\end{bmatrix} + \underline{\alpha d}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

lin. comb. of $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$

$\Rightarrow$ $\alpha\vec{v}$ is in the set V

V is closed under scalar mult.

V is a subspace of $\mathbb{R}^3$

Every vector in V is in $\mathbb{R}^3$
and V behaves like a vector space

Is the set $T = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \middle| \ t \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^3$

A: <u>No</u>

① $\vec{v}_1, \vec{v}_2$ $\quad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t_2 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} (t_1 + t_2) + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ is not in T. X

② $\vec{V} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \alpha \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \alpha t_1 + \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$ X

③ Is $\vec{0}$ in T? $\qquad \vec{0} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

has no solution (no value of t makes it possible to have the zero vector) X

T is NOT a subspace of $\mathbb{R}^3$

Columnspace (of a matrix A) $\quad A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$

$$C(A) = \underbrace{\text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}}_{\text{Definition}}$$

Not necessarily the most compact expression

↳ solve $A\vec{x} = \vec{0}$ to identify linearly independent columns (find a basis for $C(A)$)

Nullspace (of a matrix A)

$N(A)$ = Set of all $\vec{x}$ that satisfy $A\vec{x} = \vec{0}$

$$\{\vec{x} \mid A\vec{x} = \vec{0}\}$$

↳ solve $A\vec{x} = \vec{0}$ using G.E.

② ⓐ $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ What is $C(A)$? What's its dim.?

ⓘ $C(A) = \text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\} \Leftarrow$

Def.

$= \text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

Span of basis vec.

$\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ is a basis for $C(A)$

basis

$\dim(C(A)) = 1$

$A \longrightarrow \text{rref}(A) = \begin{bmatrix}\boxed{1}&0\\0&0\end{bmatrix}$

leading entries tell us which columns of orig. A are non-expressible as lin. comb. of other col.

ⓘⓘ $N(A) = ?$ $A\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c}1&0&0\\0&0&0\end{array}\right]$

$\underbrace{}_{A}$ $\underbrace{}_{\vec{0}}$

since in rref, identify free & basic variables, write solns.

$x_1 = 0 \Leftarrow 1x_1 + 0x_2 = 0$

$x_2 = \underline{t}$ (t is a real #, any)

$\vec{x} = \begin{bmatrix}0\\t\end{bmatrix} = t\begin{bmatrix}0\\1\end{bmatrix}$

using GE → $N(A) = \text{Span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$

$\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis for $N(A)$

$\dim(N(A)) = 1$

(iii) $C(A) = C(\mathrm{rref}(A))$?

$A = \mathrm{rref}(A)$ in this case, so yes.
But generally, not the same.
So generally $C(A) \neq C(\mathrm{rref}(A))$
Why? Because column entries change under row ops.

(iv) A has $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ as its col.s

Basis? $\boxed{\text{Not a basis for } \mathbb{R}^2}$

(1) Lin. indep.? $\boxed{\text{No. } \vec{0}}$

(2) Spans $\mathbb{R}^2$?

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$ $C(A) = ?$

(i) $\text{Span}\left\{\underbrace{\begin{bmatrix} -2 \\ 3 \end{bmatrix}}_{\text{basis}}\right\}$ ✓

$\dim(C(A))?$ $\boxed{1}$

$RREF(A) = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

first col. forms a basis for $C(A)$

(ii) $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ is a basis for $N(A)$

$N(A) = \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ 3 & -6 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$A\vec{x} = \vec{0}$

$x_1 = 2t$
$x_2 = t$

$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t$

$\text{Dim}(N(A)) = 1$

(iii) $A = \begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

$\text{rref}(A) = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$C(RREF(A)) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$\neq \text{Span}\left\{\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right\}$

$= C(A)$

(iv) No, columns of A are not a basis

Which condition fails?

Not linearly independent

(8) $A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

$C(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$

$\dim(C(A)) = 2$

$\text{rref}(A) \neq \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 5/2 & 1/2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 1/2 & -7/2 \\ 0 & 1 & 5/2 & 1/2 \end{bmatrix}$

$N(A)$ is a subspace of $\mathbb{R}^4$

$\dim(N(A)) = 2$

Basis vectors for $N(A)$

$s \begin{bmatrix} -1/2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$

$N(A)$ $\quad A\vec{x} = \vec{0}$

$\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & 1/2 & 0 \end{array}\right]$

$x_1 + 1/2 s - 7/2 t = 0$

$x_2 + 5/2 s + 1/2 t = 0$

$x_3 = s$

$x_4 = t$

$\Rightarrow \vec{x} = \begin{bmatrix} -1/2 s + 7/2 t \\ -5/2 s - 1/2 t \\ s \\ t \end{bmatrix}$

(iii) $C(A)$ vs. $C(\text{rref}(A))$ equal? Yes

because $C(A) = \mathbb{R}^2$, even if you row reduce, still same # of leading entries

$$C(A) = \mathbb{R}^2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \dots \right\}$$

cols of rref(A)

$\boxed{3}$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \underline{\det(A) = ad - bc}$