EECS16A

Dis 2B

① How to compute eigenvalues and eigenvectors

② Some special cases of eigenvalues/eigenvectors

geometric interpretation

③ Applying eigenvalue idea to a transition matrix system

(i) Find eigenvalues of M and eigenvectors
(ii) State if M is invertible

Start w/ finding eigenvalues

$\det(A - \lambda I) = 0$

(1) compute $\det(A - \lambda I)$
(2) Find λ's that make equal to 0

$A\vec{v} = \lambda\vec{v} \Rightarrow A\vec{v} - \lambda\vec{v} = \vec{0}$

$\vec{v} \neq \vec{0}$ is an eigenvector of A w/ eigenvalue λ

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

(a) $M = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$M - \lambda I = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}$

$\det(M - \lambda I) = (-\lambda)(-3-\lambda) - (1)(-2)$

$= \lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 1)(\lambda + 2) = 0$

Our eigenvalues for M are

$\lambda_1 = -1$

$\lambda_2 = -2$

$$M = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad \lambda_1 = -1 \quad \lambda_2 = -2$$

$$M\vec{v} = \lambda \vec{v}$$

$$(M - \lambda I)\vec{v} = \vec{0}$$

For $\lambda_1 = -1$

$$(M - \lambda_1 I)\vec{v} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \vec{v} = \vec{0}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = -t$ $\quad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$

$x_2 = t$

$\rightarrow$ $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector (λ_1)

$\rightarrow$ The eigenspace of $\lambda_1 = -1$ is span$\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$$(M - \lambda_2 I)\vec{v}_2 = \vec{0}$$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \vec{v}_2 = \vec{0}$$

By inspection, try $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \vec{0} \checkmark$$

$\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector for $\lambda_2 = -2$

$M = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ M is invertible. Why? How does that relate to eig.?

$\underline{M\vec{v} = \lambda\vec{v}}$ $\quad M\vec{v} = \vec{0}$. M has a nontrivial nullspace

$\vec{v} \neq \vec{0}$
(def. of eigenvector)

$\Updownarrow$

M has linearly dependent col.

$\Updownarrow$

M is not invertible

If $M\vec{v} = \vec{0}$

then $\lambda = 0$ has to be an eigenvalue.

M has $\lambda_1 = -1$ $\lambda_2 = -2$. Neither are 0.

[If none of the eigenvalues of a matrix are 0,
then it is invertible]

(b) Find eigval. & eigvec. of $M = \begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix}$

(1) Eigenvalues λ

$\Rightarrow \det(M - \lambda I) = 0$

$$\det\left(\begin{bmatrix} -2-\lambda & 4 \\ -4 & 8-\lambda \end{bmatrix}\right) = (-2-\lambda)(8-\lambda) - (-16) = 0$$

$$\lambda^2 - 6\lambda \cancel{-16+16} = 0$$

$$\lambda(\lambda - 6) = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = 6$$

(ii) M is not invertible bc. $\lambda_1 = 0$

(2) Find Eigenvectors $\vec{v}_1, \vec{v}_2$

For $\boxed{\lambda_1 = 0}$

$(M - \lambda_1 I)\vec{v}_1 = \vec{0}$

$\begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix} \vec{v}_1 = \vec{0}$

By inspection $\boxed{\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$ ✓

For $\boxed{\lambda_2 = 6}$

$(M - \lambda_2 I)\vec{v}_2 = \vec{0}$

$\begin{bmatrix} -8 & 4 \\ -4 & 2 \end{bmatrix} \vec{v}_2 = \vec{0}$

By inspection $\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

Q: If $\vec{v}$ is an eigvec. can $\alpha\vec{v}$ also be an eigvec? Yes.

$$A\vec{v} = \lambda\vec{v} \quad (\vec{v} \neq \vec{0},\ \alpha \neq 0)$$

$$\underline{A(\alpha\vec{v})} = \alpha(A\vec{v}) = \alpha(\lambda\vec{v}) = \lambda(\alpha\vec{v})$$

$\boxed{1}$ c : You can have repeated eigenvalues but can only get one eigenvector

$\boxed{1}$ d : You can have imaginary valued eigenvalues & eigenvectors

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \det(M - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1$$

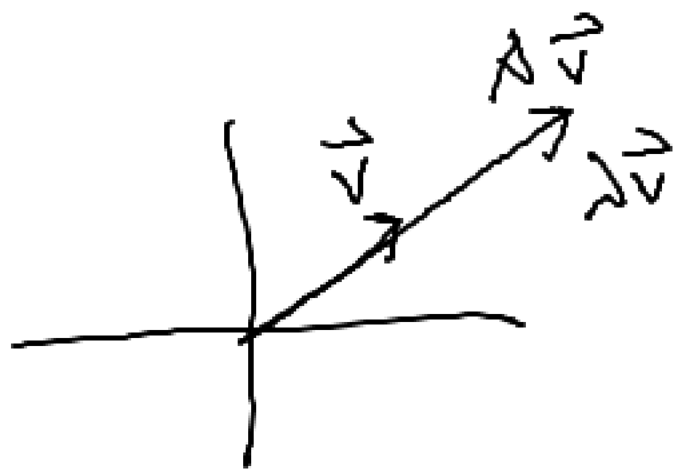
$$\lambda = \pm i$$

$(M - \lambda_2 I)$

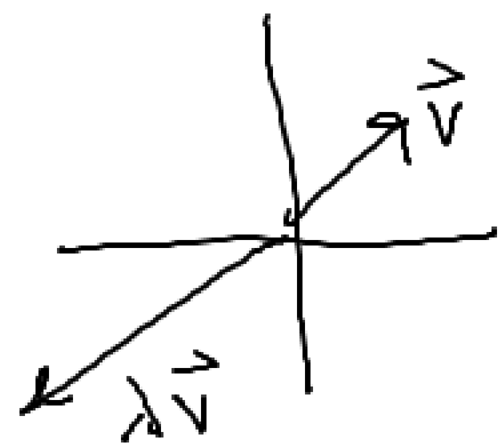
$$\lambda_1 = i \quad \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\vec{v}_1 = \vec{0} \qquad \lambda_2 = -i \quad \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\vec{v}_2 = \vec{0}$$

$$\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \underline{i\vec{v}_2 = \begin{bmatrix} i \\ -1 \end{bmatrix}} \ \checkmark$$

$\boxed{2}$ $A\vec{v} = \lambda\vec{v}$



$\lambda > 0$



$\lambda < 0$

See what happens to standard basis vec.s

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\end{bmatrix} \quad (\text{eg. } \mathbb{R}^3)$$

ⓐ I $\lambda = 1$

$(M - \lambda I)\vec{v} = \vec{0}$

$(I - I)\vec{v} = 0\underline{\vec{v}} = \vec{0}$

$I\vec{v} = 1\cdot\vec{v}$ $\Leftarrow$

Eigenvectors for I for $\lambda = 1$ all of $\mathbb{R}^n$. $\swarrow$ Can say $\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \ldots$ (choose columns of I matrix)

We don't have any other eigenvalues

(n repeated $\lambda = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\ldots$ $\lambda_n = 1$)

ⓑ Eigenspace of $\lambda = 1$ is $\mathbb{R}^n$

ⓒ Is to give a basis for the eigenspace

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

From chat

- n eigenvalues ✓
- $\lambda_1 = d_1$, $\lambda_2 = d_2$, ... ✓

Eigenvector for λ_i is i-th standard basis vector $\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th} \right)$

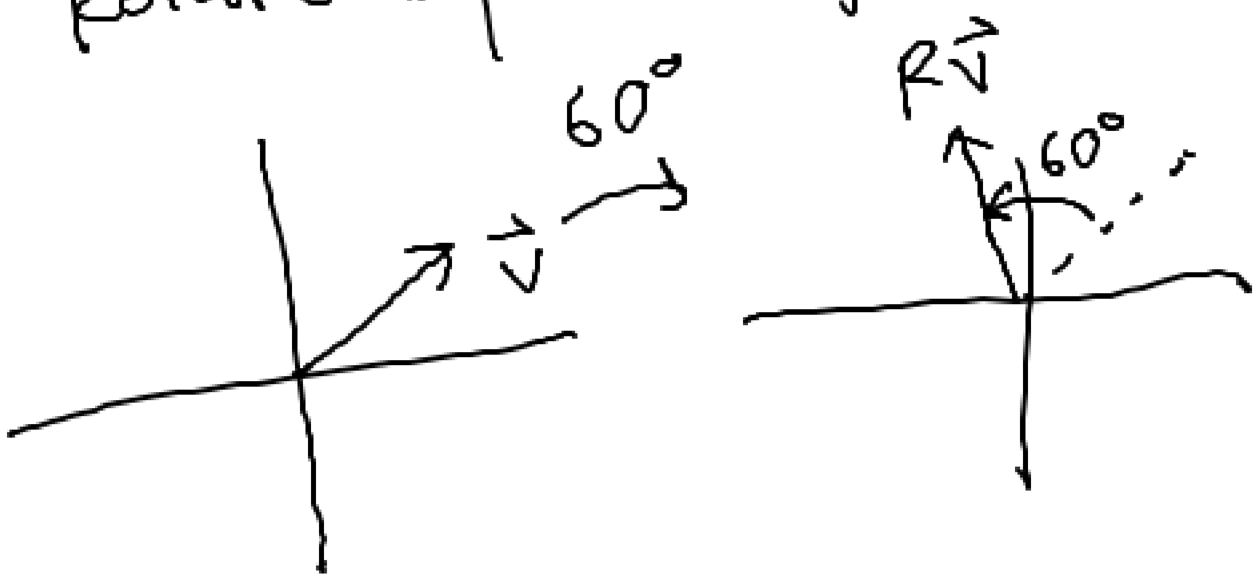
$$D \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th} = \begin{bmatrix} 0 \\ \vdots \\ d_i \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th pos.} = \underline{d_i} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\boxed{2}$ (c) Do rotations in $\mathbb{R}^2$ have eigenvalues?

Rotate by $0^\circ \Rightarrow I \Rightarrow \boxed{2}$ (a), yes $\lambda = 1$

Rotate by an angle that is not $360^\circ n$ (n is integer)

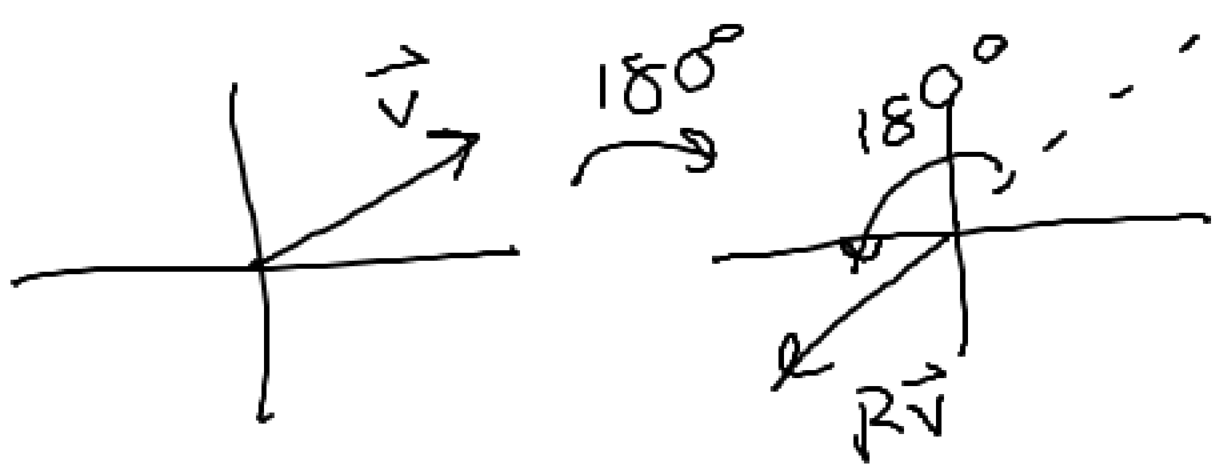
$180^\circ n$ (n is integer)



$R\vec{v}$ is a diff. direction from $\vec{v}$

There are no real eigenvalues

Rotate by 180° $\lambda = -1$



$R\vec{v} = -\vec{v}$

$\lambda = -1$

3 How to apply eigvec. eigval. to transition matrix system?

(a) $T = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix} \begin{array}{c} x_A \\ x_B \\ x_C \end{array}$

(b) Find a steady state vector $\Longleftrightarrow$ Find the eigenvector corresponding to $\lambda = 1$

$T\vec{x} = \vec{x}$

$T\vec{x} = \underset{\uparrow\, 1}{\lambda}\vec{x}$

$\underbrace{(T-I)\vec{x} = \vec{0}}$

$[\,T - I \mid \vec{0}\,]$