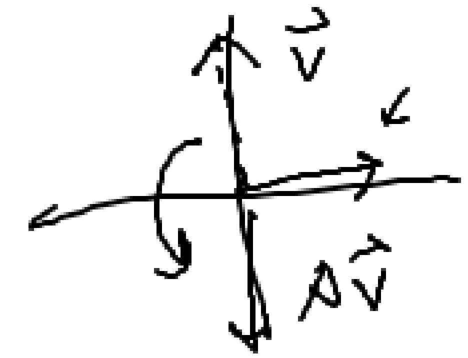
Aragorn's Odyssey (MT1, Fa19, Q6)

(a) $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ ←

A: Reflect about x-axis (negates y) ← Can you have the proper matrix order

B: Rotate CCW by 30°

$A\vec{v}$ Q: Find BA



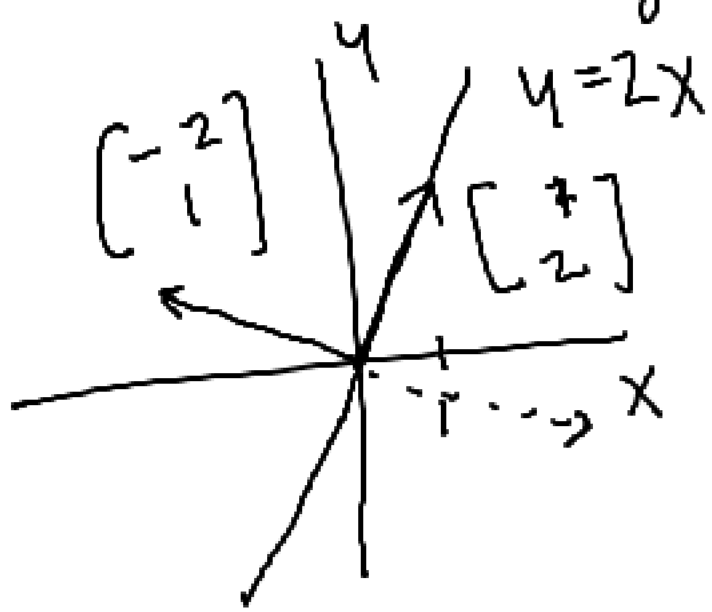
$\underbrace{BA}\vec{v}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$B = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}$ ← Can you write lin. trans. as matrices

$BA = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 1/2 \\ \frac{1}{2} & -\sqrt{3}/2 \end{bmatrix}$ ← Matrix product

Aside: Finding matrices for reflections



Find transform: reflection about $y=2x$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad A\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

How to find A? (Not $A\vec{x} = \vec{b}$)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{cases} a+2b & = 1 \\ c+2d & = 2 \\ -2a+b & = 2 \\ -2c+d & = -1 \end{cases} \longrightarrow \text{tells you } A$$

(b) Transform: reflect first ← reflections don't change length ✓

rotate first ← rotations don't change length ✓

1 unit away after operations ↑

(c) Bspell, 10 times, $\vec{x}[t+1] = B_{spell}\,\vec{x}[t]$ ⇐

$= \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$, $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ What is $\vec{x}[10]$?

$\vec{x}[1] = B\,\vec{x}[0]$

$\vec{x}[2] = B\,\vec{x}[1] = \underline{B^2\,\vec{x}[0]}$

$\vec{x}[10] = \underline{B}^{10}\,\vec{x}[0]$ ⇐ hard to do, is there insight

$\underline{\vec{x}[1] = B\,\vec{x}[0]}$

$\vec{x}[1] = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ \underline{0} \end{bmatrix}$ ←

$\underbrace{\vec{x}[1] = \underline{2\,\vec{x}[0]} \overset{?}{=} B\,\vec{x}[0]}$ ← $\vec{x}[0]$ is an eigenvector

$B\vec{v} = \lambda\vec{v}$ $\quad B^2\vec{v} = B\lambda\vec{v} = \lambda^2\vec{v}$

$\Rightarrow B^n\vec{v} = \lambda^n\vec{v}$

$B^{10}\begin{bmatrix}1\\0\end{bmatrix} = \lambda^{10}\begin{bmatrix}1\\0\end{bmatrix} = 2^{10}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1024\\0\end{bmatrix}$

$\lambda = ?$ $\quad \lambda = 2$

(d) $\vec{x}[t]$, $\vec{0}$ (start point) $\quad \vec{x}[0] = \vec{0}$ $\quad C_{spell} \neq 0_{n\times n}$

$\vec{x}[t+1] = C_{spell}\,\vec{x}[t] + \vec{b}\,u[t]$

- Ship can reach $\mathrm{Span}\{\vec{b}, C\vec{b}, C^2\vec{b}, \ldots, C^9\vec{b}\}$
- $\vec{b} \neq 0$ is an eigenvector (of C) $\rightarrow$ $\boxed{C\vec{b} = \lambda\vec{b}}$

What is the dim. of the space you can reach?

$\Rightarrow \dim(\mathrm{Span}\{\underbrace{\vec{b}, C\vec{b}, C^2\vec{b}, \ldots, C^9\vec{b}}_{\text{Not necessarily a basis}}\}) \rightarrow \dim(\mathrm{Span}\{\vec{b}, \lambda\vec{b}, \ldots, \lambda^9\vec{b}\})$

$\hookrightarrow \dim(\mathrm{Span}\{\vec{b}\}) = \boxed{1}$

Trouble in Telecomm (MT 1, Fa18, q4)

(a) $N(V_0) \to$ Find a basis for $N(V_0)$

How? $A\vec{x} = \vec{0} \Rightarrow$ $\boxed{\text{Row reduce / GE on } V_0}$

on $A = V_0$ Find $\vec{x}$ that $V_0\vec{x} = \vec{0}$

$$\left[\begin{array}{c|c} V_0 & \begin{matrix}0\\0\\0\end{matrix}\end{array}\right]$$

nothing happens to 0's

$$\to \begin{bmatrix}1&0&2\\2&3&10\\0&2&4\end{bmatrix} \to \begin{bmatrix}1&0&2\\0&3&6\\0&2&4\end{bmatrix} \to \begin{bmatrix}1&0&2\\0&1&2\\0&2&4\end{bmatrix} \to \begin{bmatrix}1&0&2\\0&1&2\\0&0&0\end{bmatrix} \to \left[\begin{array}{ccc|c}1&0&2&0\\0&1&2&0\\0&0&0&0\end{array}\right]$$

Once you've finished w/ a prob. try to check your answer in a way diff. than you've solved it. Check: $V_0\begin{bmatrix}-2\\-2\\1\end{bmatrix} \stackrel{?}{=} \vec{0}$

$x_3 = s$

$x_1 = -2s$

$x_2 = -2s$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2\\-2\\1\end{bmatrix}s$$

$\boxed{\text{Basis : } \left\{\begin{bmatrix}-2\\-2\\1\end{bmatrix}\right\}}$

(5) $V_0 = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 10 \\ 0 & 2 & 4 \end{bmatrix}$ is it invertible?

$\boxed{\text{No!}}$ Non-trivial nullspace / $\det(V_0) = 0$ / lin. dep col.

$V_0 \vec{x} = \vec{0}$

$V_0 \vec{x}_1 = \vec{y}_1$

$V_0 (\vec{x} + \vec{x}_1) = \vec{y}_1$

Can't tell where we started from (two diff. vectors give us the same thing)

[Note the relationships between invertibility, lin dep., and other concepts]

V_0 is a bad encoding matrix (we can't decode)

[reread problem prompt JIC]

[a set is necessarily Lin Dep. if it has $\vec{0}$]

$\det(V_0) = 0$ While true, we've not shown 3x3 det.

(c) $V_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is it invertible? Good? Bad?

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$$

$$V_1^{-1} = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

$V_1 V_1^{-1} = \underline{I}$ ✓ then we're good.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Good ✓

Because we can decode all messages