EECS 16A DIS 6A

Topics from lecture that will show up

- inner product (of vectors)

Takeaways

- Conceptual interpretation of inner products (more on this tomorrow aswell)
- How to compute the standard inner product
- Signals: what are they, notation
- Cross correlation: shifted inner products, how to compute

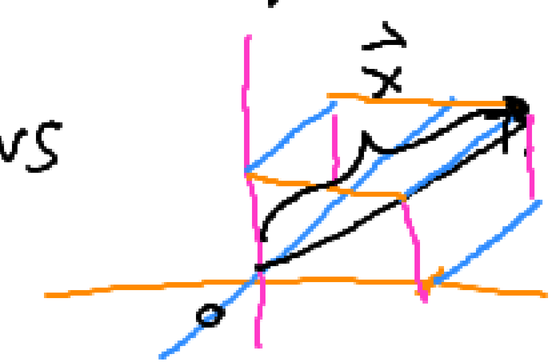
Inner products: mapping of two vectors to a real number

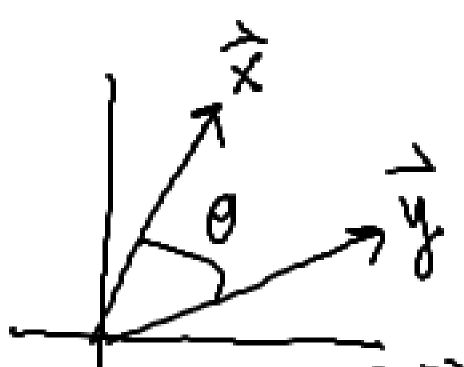
$\langle \vec{x}, \vec{x} \rangle$ (inner product of $\vec{x}$ with itself)

$\hookrightarrow$ length (magnitude / norm)

$\langle \vec{x}, \vec{y} \rangle$ (inner product of $\vec{x}$ with $\vec{y}$)

$\hookrightarrow$ angle between the two vectors





$\theta \sim \langle \vec{x}, \vec{y} \rangle$

1. How to compute inner product?

Vectors in $\mathbb{R}^n$ → the standard inner product $\underline{\vec{x}^T\vec{y}} = \underbrace{\langle \vec{x}, \vec{y} \rangle}$

($\vec{x}, \vec{y} \in \mathbb{R}^n$) $\quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = [x_1 \cdots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots + x_n y_n$$
$$= \underline{\sum_{i=1}^{n} x_i y_i}$$

(a) $\vec{x}, \vec{y} \in \mathbb{R}^3$

$\vec{x} = \begin{bmatrix} \boxed{1} \\ \boxed{0} \\ \boxed{1} \end{bmatrix}, \vec{y} = \begin{bmatrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \end{bmatrix}$

$\langle \vec{x}, \vec{y} \rangle = \boxed{1} \cdot \boxed{1} + \boxed{0} \cdot \boxed{2} + 1 \cdot 3$
$= 4$

$\left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\rangle$

(b) $\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle = 4$ (example that shows symmetry)

(c) $\left\langle \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\rangle = 0$

$1 \cdot (-3) + 0 \cdot 2 + (3)(1)$

Q: What does it mean if $\langle \vec{x}, \vec{y} \rangle = 0$?

A: says something about angle

Inner products have to satisfy

(a) Symmetry → swap vectors, same inner product

$$\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$$

(b) Linearity → taking inner products with a linear combinations of vectors will be a linear combination of inner products

$$\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$$

$$\langle \underbrace{c}_{\text{scalar}}\vec{x}, \vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$$

in general

$$\langle \vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_n, \vec{y} \rangle = \langle \vec{x}_1, \vec{y} \rangle + \langle \vec{x}_2, \vec{y} \rangle + \cdots$$

(c) Positive definiteness → an inner product of a vector with itself is positive, unless you have the zero vector → inner product is 0.

$$\langle \vec{x}, \vec{x} \rangle > 0 \text{ if } \vec{x} \neq \vec{0} \qquad \langle \vec{0}, \vec{0} \rangle = 0$$

Euclidean inner product - same as standard inner product

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y}$$

(a) How to show symmetry for $\langle \vec{x}, \vec{y} \rangle = \underline{\vec{x}^T y}$ in $\mathbb{R}^2$? (for all $\vec{x}, \vec{y} \in \mathbb{R}^2$)

Suggestion: • try writing $\langle \vec{x}, \vec{y} \rangle$ and $\langle \vec{y}, \vec{x} \rangle$

• Pick two vectors in $\mathbb{R}^2$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = [x_1 \; x_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + x_2 y_2$

$\langle \vec{y}, \vec{x} \rangle = \vec{y}^T \vec{x} = [y_1 \; y_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y_1 x_1 + y_2 x_2$

are they equal?

scalar multiplication is commutative

the two expressions above are =

$\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$ (yes, symmetric)

$\boxed{2}$ (b) Linearity: is $\vec{x}^T\vec{y}$ linear?

$\left(\underbrace{(c\vec{x})^T\vec{y}} = \underbrace{c(\vec{x}^T\vec{y})}^{?},\quad (\vec{x}_1+\vec{x}_2)^T\vec{y} = \vec{x}_1^T\vec{y} + \vec{x}_2^T\vec{y}\ ? \right)$

- Can use same approach as before ←
- Alternative approach: Matrix-matrix multiplication

$\boxed{\vec{x}^T\vec{y} \text{ is linear}}$

$$(c\vec{x})^T = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}^T = [cx_1 \cdots cx_n] = c(x_1 \cdots x_n) = c\vec{x}^T$$

$$(c\vec{x})^T\vec{y} = \underline{c}(\vec{x}^T\vec{y}) \checkmark$$

$$(\vec{x}_1+\vec{x}_2)^T = \begin{bmatrix} x_{11}+x_{21} \\ \vdots \\ x_{1n}+x_{2n} \end{bmatrix}^T = [x_{11}+x_{21} \cdots x_{1n}+x_{2n}]$$

$$= [x_{11} \cdots x_{1n}] + [x_{21} \cdots x_{2n}]$$

$$= \vec{x}_1^T + \vec{x}_2^T$$

$$\underbrace{(\vec{x}_1+\vec{x}_2)^T\vec{y}} = (\vec{x}_1^T+\vec{x}_2^T)\vec{y} = \vec{x}_1^T\vec{y} + \vec{x}_2^T\vec{y} \checkmark$$ (distributive prop. of matrix-matrix mult.)

matrix-matrix mult.

Signals

Signals are functions that capture information / contain information

examples
- → voltage as a function of time
- → current as a function of time
- → sand level / loudness a function of time

Assume discrete time — time is in integers

Notation: $s[n]$ n is a integer , $s[n]$ is a real number for the value of the signal at time n.

s ← signal variable name

n ← time variable name

$s[0]$ → value of signal at time zero

$s[-3]$ → value of signal at time -3

Signals are like infinitely long vectors :

$$\begin{bmatrix} \vdots \\ s[-1] \\ s[0] \\ s[1] \\ s[2] \\ \vdots \end{bmatrix}$$

inner product ~ signal

$\langle s[n], s[n] \rangle$ ~ amount of energy (e.g. louder music)

$\Rightarrow \langle s_1[n], s_2[n] \rangle$ ~ similarity

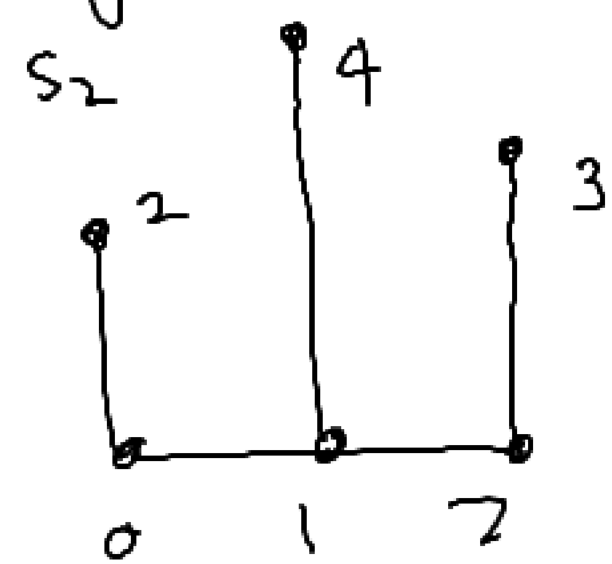
Cross correlation (of two signals) → takes two signals and gives another signal

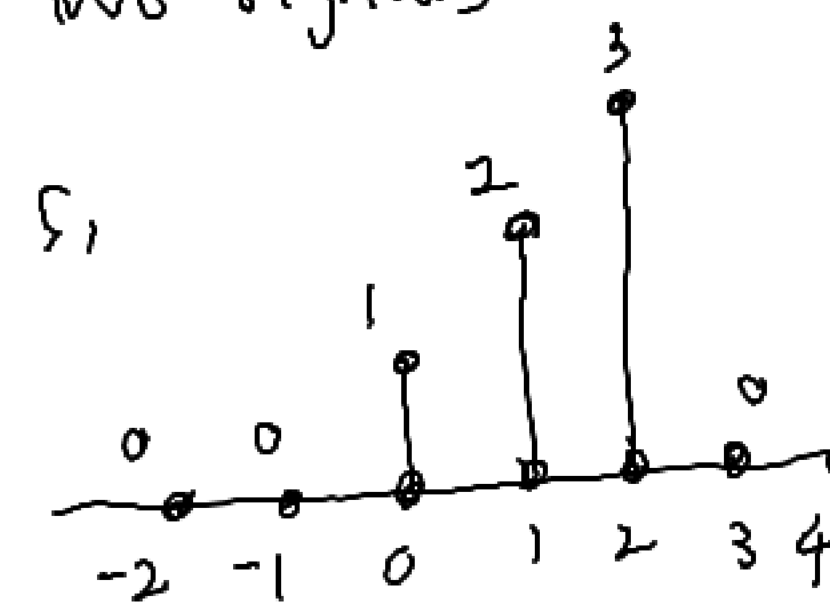
$$\mathrm{corr}_{s_1}(s_2)[k] = \underbrace{\sum_{n=-\infty}^{\infty} s_1[n]\, s_2[n-k]}_{\text{"shifted inner product for infinitely long vectors"}}$$

shifted by k

Example of computing cross correlation of two signals

(3)





$s_1[0] = 1$

$s_1[1] = 2$

$s_1[2] = 3$

$s_2[0] = 2$

$s_2[1] = 4$

$s_2[2] = 3$

$s_1[-1] = ?$

$s_1[3] = ?$

Assume that values not given are 0

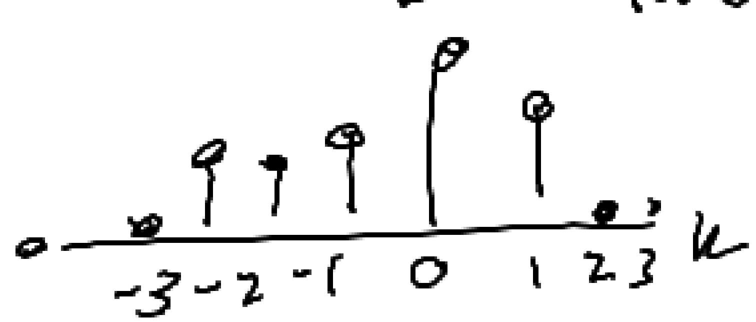
$\to n$	-3	-2	-1	0	1	2	3
$s_1[n]$	0	0	0	1	2	3	0
$s_2[n+2]$	0	2	4	3	0	0	0

← shifted back by 2

$\langle s_1[n], s_2[n+2] \rangle$: $0 \cdot 0 + 2 \cdot 0 + 4 \cdot 0 + 3 \cdot 1 + 0 + 0 + 0$

$\mathrm{corr}_{s_1}(s_2)[2] = 3$

$\mathrm{corr}_{s_1}(s_2)[k]$ might look like



n	-3	-2	-1	0	1	2	3
$s_2[n+1]$	0	0	2	4	3	0	0 …