EECS16A DIS 7B

- OMP Example : OMP as iterated least squares
- Past final problem with OMP

* Discussion checkoff (if you still need it)

$M\vec{x} = \vec{y}$

known known

$\vec{x} \to$ want to find

$$\begin{array}{cc} & 4 \\ 3 & \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \\ & M \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \approx \underbrace{\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}}_{\vec{y}}$$

$\boxed{\text{wide}}\ \vec{x} = \vec{b}$

$A\vec{x} \approx \vec{b}$

(a) Why is $\vec{x}$ not solvable for directly?

GE , LS

(b) LS: $\hat{x} = \boxed{(M^TM)^{-1}} M^T\vec{y}$

Not computable

$M \to$ lin indep cols? No

$rk(M) \leq 3$

$M^TM \to 4\times4$

$\hookrightarrow$ isn't invertible

GE: free variable - might get infinitely many sol.

or Inconsistent system

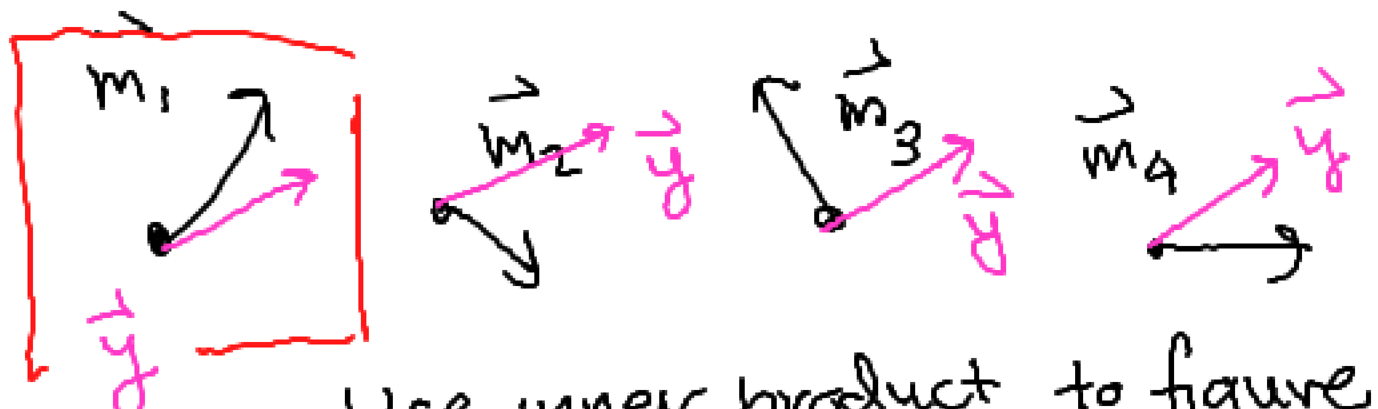
$\to$ no sol.

© OMP algorithm

$M, \vec{y}$ Find $\hat{x}$

$$M = \begin{bmatrix} | & & | \\ \vec{m}_1 & \dots, & \vec{m}_4 \\ | & & | \end{bmatrix} \qquad \vec{e}_0 = \vec{y}$$

①



Use inner product to figure out which vector is closest (largest $|\langle \vec{m}_i, \vec{y} \rangle|$)

② Do least squares on chosen $\vec{m}_i$ ($\vec{m}_1$)

$$M_1 \hat{x}_1 = \begin{bmatrix} \vec{m}_1 \end{bmatrix} \hat{x}_1 \approx \begin{bmatrix} \vec{y} \end{bmatrix} \to \vec{m}_1 \hat{x}_1 \to \hat{y}_1$$

Least sq. works because $\vec{m}_1$ is linindep

③ Compute error from est. and try to explain error with other vectors

$\hat{y}_1$ - first est. of $\vec{y}$

$\vec{e}_1$ - error

$\vec{y} - \hat{y}_1$

④ $M_2 \hat{x}_2 = \begin{bmatrix} | & | \\ \vec{m}_1 & \vec{m}_2 \\ | & | \end{bmatrix} \hat{x}_2 \approx \vec{y}$

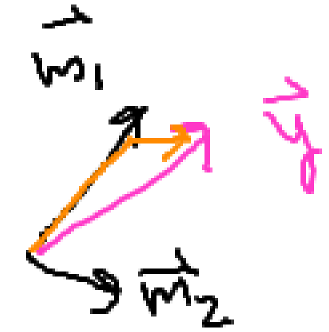
Iterating

Q: Do we choose lin dep col at any point?

A: No

Doing 1st sqs with $\begin{bmatrix} | & | \\ \vec{m}_1 & \vec{m}_2 \\ | & | \end{bmatrix} \vec{x}_2 \approx \vec{y}$

$\vec{e}_2 = \vec{0}$



We can stop when

① Hit a certain # of non-zero entries (sparsity condition)

② Error magnitude is acceptably small

① Compute inner products

$$\left\langle \underset{\vec{y}}{\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{m}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \right\rangle = 5$$

$$\left\langle \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\rangle = 3$$

$$\left\langle \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \right\rangle = 12 \qquad \boxed{\vec{m}_3}$$

$$\left\langle \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle = 6$$

$$\vec{m}_3 \hat{x}_1 \approx \vec{y}$$

$$\hat{x}_1 = \left(\vec{m}_3^T \vec{m}_3\right)^{-1} \vec{m}_3^T \vec{y}$$

$$= \left\langle \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\rangle^{-1} (12)$$

$$= 8^{-1}(12)$$

$$= \frac{12}{8} = \frac{3}{2}$$

$$\vec{x} \approx \begin{bmatrix} 0 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

$$\vec{e}_1 = \vec{y} - \hat{y}_1$$

$$= \vec{y} - \vec{m}_3 \hat{x}_1$$

$$= \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \frac{3}{2}$$

$$= \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\langle \vec{e}_1, \vec{m}_1 \rangle =$$

$$\left\langle \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\rangle = 0$$

Compute inner products

$\vec{m}_1$

$$\left\langle \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle = 2$$

$\vec{m}_2$

$$\left\langle \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\rangle = 0$$

$\vec{m}_4$

$$\left\langle \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle = \underline{3}$$

$$\begin{bmatrix} \vec{m}_3 & \vec{m}_4 \end{bmatrix} \hat{x}_2 \approx \vec{y}$$

M_2

$$\begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \hat{x}_2 \approx \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{x}_2 = (M_2^T M_2)^{-1} M_2^T \vec{y}$$

$$= (M_2^T M_2)^{-1} \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$

$$= (M_2^T M_2)^{-1} \begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} \hat{x}_{21} \\ \hat{x}_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 1 \\ -2 + 4 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$\boxed{\vec{x} \approx \begin{bmatrix} 0 \\ 0 \\ \hat{x}_{21} \\ \hat{x}_{22} \end{bmatrix}}$$

stop here because 2 non-zero entries

$$M_2^T M_2 = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 \\ 2 & 2 \end{bmatrix}$$

$$(M_2^T M_2)^{-1} = \frac{1}{16-4} \begin{bmatrix} 2 & -2 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} \end{bmatrix}$$

$$\hat{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\hat{y}_2 = M_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Q: what if $\vec{m}_1 = \alpha \vec{m}_3 + \beta \vec{m}_4$

A: $\begin{cases} \langle \vec{e}, \vec{m}_3 \rangle = 0 \\ \langle \vec{e}, \vec{m}_4 \rangle = 0 \end{cases}$

$$\Rightarrow \langle \vec{e}, \vec{m}_1 \rangle = \langle \vec{e}, \alpha \vec{m}_3 + \beta \vec{m}_4 \rangle$$
$$= \alpha \langle \vec{e}, \vec{m}_3 \rangle + \beta \langle \vec{e}, \vec{m}_4 \rangle$$
$$= 0$$

Caveat: we're using inner product as a measure of similarity

Q: Inner product is not a good measure of sim?

A: Yeah - want to normalize

Want: columns of M (signals) to be normalized or all same norm

$\boxed{2}$ OMP problem - Fall '15 final

Don't know our numerical vectors (M, $\vec{b}/\vec{y}$)

→ Inner products

→ Can we still do OMP?

(a) Which of $\vec{c}_1, \ldots, \vec{c}_5$ has largest inner prod. w/ $\vec{b}$?

$\langle \vec{c}_1, \vec{b} \rangle = 1$

$\langle \vec{c}_2, \vec{b} \rangle = -5$

$\vdots$

$\boxed{\langle \vec{c}_4, \vec{b} \rangle = 6}$ → select $\vec{c}_4$ to do OMP with

(b) $\vec{c}_a = \vec{c}_4$ $\quad$ $\underline{\vec{b}_p} = \text{proj}_{\vec{c}_a} \vec{b} = \text{proj}_{\vec{c}_4} \vec{b} = \boxed{\frac{\overbrace{\langle \vec{b}, \vec{c}_4 \rangle}^{6}}{\underbrace{\langle \vec{c}_4, \vec{c}_4 \rangle}_{2}} \vec{c}_4} = \underline{3\vec{c}_4}$

(c) residue - error

$\vec{e} = \vec{b} - \vec{b_p}$

$\langle \vec{e}, \vec{c_i} \rangle = \langle \vec{b}, \vec{c_i} \rangle - \langle \vec{b_p}, \vec{c_i} \rangle$

$= \langle \vec{b}, \vec{c_i} \rangle - \langle 3\vec{c_4}, \vec{c_i} \rangle$

$= \langle \vec{b}, \vec{c_i} \rangle - 3 \langle \vec{c_4}, \vec{c_i} \rangle$

Choose $\vec{c_1}$ next

$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$1-3(-1)$	$-5-3(-1)$	$2-3(0)$	$6-3(2)$	$-1-3(-1)$
$\boxed{4}$	-2	2	0	2

(d) cont

$$\hat{x} = (M_2^T M_2)^{-1} M_2^T \vec{b} \qquad \hat{x} = \begin{bmatrix} \hat{x}_a \\ \hat{x}_c \end{bmatrix}$$

$$= \left(\begin{bmatrix} | & | \\ \vec{c}_4 & \vec{c}_1 \\ | & | \end{bmatrix}^T \begin{bmatrix} | & | \\ \vec{c}_4 & \vec{c}_1 \\ | & | \end{bmatrix} \right)^{-1} \begin{bmatrix} | & | \\ \vec{c}_4 & \vec{c}_1 \\ | & | \end{bmatrix}^T \vec{b}$$

$$= \left(\begin{bmatrix} - & \vec{c}_4^{\,T} & - \\ - & \vec{c}_1^{\,T} & - \end{bmatrix} \begin{bmatrix} | & | \\ \vec{c}_4 & \vec{c}_1 \\ | & | \end{bmatrix} \right)^{-1} \begin{bmatrix} \vec{c}_4^{\,T} \vec{b} \\ \vec{c}_1^{\,T} \vec{b} \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} \vec{c}_4^{\,T} \vec{c}_4 & \vec{c}_4^{\,T} \vec{c}_1 \\ \vec{c}_1^{\,T} \vec{c}_4 & \vec{c}_1^{\,T} \vec{c}_1 \end{bmatrix}^{-1} \begin{bmatrix} \vec{c}_4^{\,T} \vec{b} \\ \vec{c}_1^{\,T} \vec{b} \end{bmatrix} = \begin{bmatrix} \hat{x}_a \\ \hat{x}_c \end{bmatrix}}$$

(e) $\Rightarrow$ sub in #s, solve

Q: when to not use OMP

→ when not a sparse prob.