EECS 16A Designing Information Devices and Systems I
Summer 2020 Homework 1A

This homework is due Wednesday, July 01, 2020 at 23:59 PT.
Self-grades are due Sunday, July 05, 2020 at 23:59 PT.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

Please attach a PDF of your Jupyter notebook for all the problems that involve coding. Make sure the results of your plots (if any) are visible. Please assign the PDF of the notebook to the correct problems on Gradescope — we will be unable to grade the problems without this assignment or submission.

Homework Learning Goals: *The objective of this homework is to introduces matrix-vector notation for a system of linear equations. Additionally, this homework introduces mathematical proofs.*

1. **Kinematic Model for a Simple Car**

Learning Goal: Many real world systems are not actually linear and have more complex behaviors. However, linear models can approximate non linear systems under certain conditions.

Building a self-driving car first requires understanding the basic motions of a car. In this problem, we will explore how to model the motion of a car.

There are several models that we can use to model the motion of a car. Assume we use a kinematic model, described in the following four equations and Figure 1.

$$x[k+1] = x[k] + v[k]\cos(\theta[k])\Delta t \tag{1}$$

$$y[k+1] = y[k] + v[k]\sin(\theta[k])\Delta t \tag{2}$$

$$\theta[k+1] = \theta[k] + \frac{v[k]}{L}\tan(\phi[k])\Delta t \tag{3}$$

$$v[k+1] = v[k] + a[k]\Delta t \tag{4}$$

where

- k, a nonnegative integer, indicates the time step at which we measure each variable (e.g. $v[k]$ is the speed at time step k and $v[k+1]$ is the speed at the following time step)
- $x[k]$ and $y[k]$ denote the coordinates of the vehicle (meters)
- $\theta[k]$ denotes the heading of the vehicle, or the angle with respect to the x-axis (radians)
- $v[k]$ is the speed of the car (meters per second)
- $a[k]$ is the acceleration of the car (meters per second squared)
- $\phi[k]$ is the steering angle input we command (radians)
- Δt is a constant measuring the time difference (in seconds) between time steps $k+1$ and k
- L is a constant and is the length of the car (in meters)

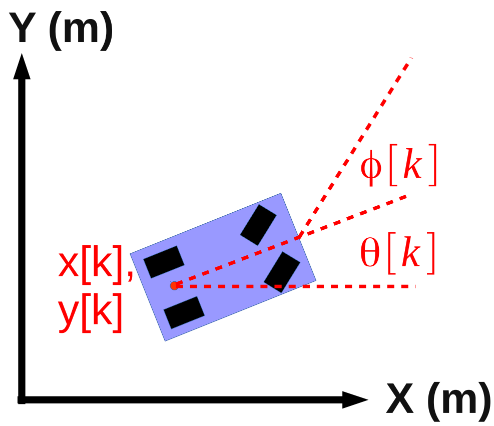

Figure 1: Vehicle Kinematic Model

For this problem, let L be 1.0 meter and Δt be 0.1 seconds.

The variables $x[k], y[k], \theta[k], v[k]$ describe the **state** of the car at time step k. The state captures all the information needed to fully determine the current position, speed, and heading of the car. The **inputs** at time step k are $a[k]$ and $\phi[k]$. These are provided by the driver. The current value of these inputs, along with the current state of the vehicle, will determine the state of the vehicle at the next time step.

We note that the problem is nonlinear, due to the sine, cosine and tangent functions, as well as terms including the product of states and inputs.

The purpose of this problem is to show that we can approximate a nonlinear model with a simple linear model and do reasonably well. This is why, despite many systems being nonlinear, linear algebra tools are widely used in practice.

For Parts (b) - (d), fill out the corresponding sections in prob1A.ipynb.

(a) We assume that the car has a small heading ($\theta \approx 0$) and that the steering angle is also small ($\phi \approx 0$), where $\approx$ means "approximately equal to." In this case, we could use the following approximations:

$$\sin(\alpha) \approx 0,$$
$$\cos(\alpha) \approx 1,$$
$$\tan(\alpha) \approx 0.$$

where α is the small angle of interest. Here, we use a very simple approximation for small angles; in later classes, you may learn better approximations.

Draw, by hand, graphs of of $\sin(\alpha)$ and $\cos(\alpha)$, for α ranging from $-\pi$ to π. Using these graphs can you justify the approximation we are making for small values of α?

(b) Applying the approximation described in the previous part, write down a system of linear equations that approximates the nonlinear vehicle model given above in Equations (1) to (4). In particular, find the 4 x 4 matrix **A** and 4 x 2 matrix **B** that satisfy the equation given below.

$$\begin{bmatrix} x[k+1] \\ y[k+1] \\ \theta[k+1] \\ v[k+1] \end{bmatrix} = \mathbf{A} \begin{bmatrix} x[k] \\ y[k] \\ \theta[k] \\ v[k] \end{bmatrix} + \mathbf{B} \begin{bmatrix} a[k] \\ \phi[k] \end{bmatrix}$$

Hint: *Start with simplifying Equations* (1) *to* (4).

(c) Suppose we drive the car so that the direction of travel is aligned with the x-axis, and we are driving nearly straight, i.e. the steering angle is $\phi[k] = 0.0001$ radians. (Driving exactly straight would have the steering angle $\phi[k] = 0$ radians.) The initial state and input are:

$$\begin{bmatrix} x[0] \\ y[0] \\ \theta[0] \\ v[0] \end{bmatrix} = \begin{bmatrix} 5.0 \\ 10.0 \\ 0.0 \\ 2.0 \end{bmatrix}$$
$$\begin{bmatrix} a[k] \\ \phi[k] \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.0001 \end{bmatrix}$$

You can use these values in the IPython notebook to compare how the nonlinear system evolves in comparison to the linear approximation that you made. The IPython notebook simulates the car for ten time steps. Are the trajectories similar or very different? Why?

(d) Now suppose we drive the vehicle from the same starting state, but we turn left instead of going straight, i.e. the steering angle is $\phi[k] = 0.5$ radians. The initial state and input are:

$$\begin{bmatrix} x[0] \\ y[0] \\ \theta[0] \\ v[0] \end{bmatrix} = \begin{bmatrix} 5.0 \\ 10.0 \\ 0.0 \\ 2.0 \end{bmatrix}$$
$$\begin{bmatrix} a[k] \\ \phi[k] \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix}$$

You can use these values in the IPython notebook to compare how the nonlinear system evolves in comparison to the linear approximation that you made. The IPython notebook simulates the car for ten time steps. Are the trajectories similar or very different? Why?

2. Linear Dependence in a Square Matrix

(a) Suppose Gaussian elimination is applied to a matrix A, and the resulting matrix (in row reduced echelon form) has at least one row of all zeros. Argue that this means that the rows of A are linearly dependent.

(b) Let A be a square $n \times n$ matrix, (i.e. both the columns and rows are vectors in $\mathbb{R}^n$). Suppose we are told that the columns of A are linearly dependent. Prove, then, that the rows of A must also be linearly dependent. You can use the conclusion from part (a) in your proof.

(**Hint**: Can you use the linear dependence of the columns to say something about the number of solutions to $A\vec{x} = \vec{0}$? How does the number of solutions relate to the result of Gaussian elimination?)

(c) Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, show the following:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

3. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?