EECS 16A Designing Information Devices and Systems I
Summer 2020 Homework 2A

This homework is due Wednesday July 8, 2020, at 23:59 PT.
Self-grades are due Sunday July 12, 2020, at 23:59 PT.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any hand-written answers should be scanned). This homework does not contain any IPython components.

Homework Learning Goals:*You should get practice with computing nullspaces and columnspaces, which is then applicable to computing eigenvalues and eigenvectors. Each of these concepts provides an important characterization of matrices.*

1. **Finding Null Spaces and Column Spaces**

   Learning Objectives*: Null spaces and column spaces are two fundamental vector spaces associated with matrices and they describe important attributes of the transformations that these matrices represent. This problem explores how to find and express these spaces.*

   Definition (Null space): The null space of a matrix, $A \in \mathbb{R}^{m\times n}$, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{0}$. The null space is notated as $N(A)$ and the definition can be written in set notation as:

   $$N(A) = \{\vec{x} \mid A\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n\}$$

   Definition (Column space): The column space of a matrix, $A \in \mathbb{R}^{m\times n}$, is the set of all vectors $A\vec{x} \in \mathbb{R}^m$ for all choices of $\vec{x} \in \mathbb{R}^n$. Equivalently, it is also the span of the set of A's columns. The column space can be notated as $\mathrm{C}(A)$ or $\mathrm{range}(A)$ and the definition can be written in set notation as:

   $$\mathrm{C}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

   (a) Consider matrices in $\mathbb{R}^{3\times 5}$. What is the maximum possible number of linearly independent column vectors?

   (b) You are given the following matrix $\mathbf{A}$.

   $$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   Find a set of vectors that span the column space of $\mathbf{A}$. What is the minimum number of vectors required to span the column space of $\mathbf{A}$? (This is the dimension of the column space of $\mathbf{A}$.)

   (c) The dimension of the null space is the minimum number of vectors needed to span it. Find a set of vectors that span the null space of $\mathbf{A}$ (the matrix from part (b)). What is the dimension of the null space of $\mathbf{A}$?

   (d) What do you notice about the sum of the dimensions of the null space and the column space in relation to the dimensions of $\mathbf{A}$?

(e) Now consider the new matrix, $\mathbf{B} = \mathbf{A}^T$,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

Find a set of vectors that span the column space of $\mathbf{B}$. What is the minimum number of vectors required to span the column space of $\mathbf{B}$?

(f) Find a set of vectors that spans the null space of the following matrix. This problem requires systematic calculations, but is helpful if you want more practice.

$$\mathbf{C} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

(g) Find the column space and its dimension, and the nullspace and its dimension of the following matrix.

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

2. Introduction to Eigenvalues and Eigenvectors

Learning Goal: *Practice algorithmic computation of eigenvalues and eigenvectors. The importance of eigenvalues and eigenvectors will become clear in the following problems.*

For each of the following matrices, find their eigenvalues and the corresponding eigenvectors. For simple matrices, you may do this my inspection if you prefer.

(a) $\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(d) Let $A \in \mathbb{R}^{n \times n}$ be a general square matrix. Show that the set of eigenvectors corresponding to a particular eigenvalue of this matrix

$$\{\vec{x} \in \mathbb{R}^n : A\vec{x} = \lambda \vec{x}, \lambda \in \mathbb{R}\}$$

is a subspace.

3. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?