EECS 16A Designing Information Devices and Systems I
Summer 2020 Homework 2B

This homework is due Sunday July 12, 2020, at 23:59 PT.
Self-grades are due Wednesday July 15, 2020, at 23:59 PT.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any hand-written answers should be scanned) as well as your IPython notebook saved as a PDF.

Please attach a PDF of your Jupyter notebook for all the problems that involve coding. Make sure the results of your plots (if any) are visible. Please assign the PDF of the notebook to the correct problems on Gradescope — we will be unable to grade the problems without this assignment or submission.

Homework Learning Goals: *This homework is all about applications for the linear algebra tools we have developed in the course so far.*

1. **Traffic Flows (PRACTICE)**

   Learning Objective: *The learning objective of this problem is to see how the concept of nullspaces can be applied to flow problems.*

   Your goal is to measure the flow rates of vehicles along roads in a town. It is prohibitively (too) expensive to place a traffic sensor along every road. You realize, however, that the number of cars flowing into an intersection must equal the number of cars flowing out. You can use this "flow conservation" to determine the traffic along all roads in a network by measuring the flow along only some roads. In this problem, we will explore this concept.

   (a) Let's begin with a network with three intersections, A, B and C. Define the flow t_1 as the rate of cars (cars/hour) on the road between B and A, flow t_2 as the rate on the road between C and B, and flow t_3 as the rate on the road between C and A.

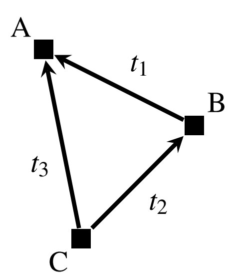

Figure 1: A simple road network.

   (Note: The directions of the arrows in the figure are the way that we define positive flow by convention. For example, if there were 100 cars per hour traveling from A to C, then $t_3 = -100$. The flows now are not fractions of water in reservoirs as in the pumps setting, but numbers of cars.)

   We assume the "flow conservation" constraints: the net number of cars per hour flowing into each intersection is zero. For example at intersection B, we have the constraint $t_2 - t_1 = 0$. The full set of constraints (one per intersection) is:

$$\begin{cases} t_1 + t_3 = 0 \\ t_2 - t_1 = 0 \\ -t_3 - t_2 = 0 \end{cases}$$

As mentioned earlier, we can place sensors on a road to measure the flow through it, but we have a limited budget, and we would like to determine all of the flows with the smallest possible number of sensors.

Suppose for the network above we have one sensor reading, $t_1 = 10$. Can we figure out the flows along the other roads? (That is, the values of t_2 and t_3). If we can, find the values of t_2 and t_3.

(b) Now suppose we have a larger network, as shown in Figure 2.

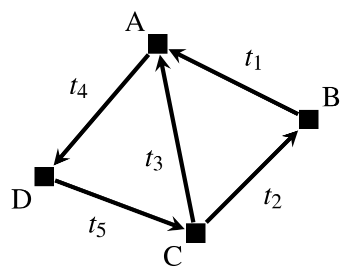

Figure 2: A larger road network.

We would again like to determine the traffic flows on all roads, using measurements from some sensors. A Berkeley student claims that we need two sensors placed on the roads CA (measuring t_3) and DC (measuring t_5). A Stanford student claims that we need two sensors placed on the roads CB (measuring t_2) and BA (measuring t_1). Write out the system of linear equations that represents this flow graph. Is it possible to determine all traffic flows, $\left[t_1, t_2, t_3, t_4, t_5\right]^T$, with the Berkeley student's suggestion? How about the Stanford student's suggestion?

(c) We would like a more general way of determining the possible traffic flows in a network. Suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. As a first step, let us try to write all the flow conservation constraints (one per intersection) as a matrix equation.

Construct a 4×5 matrix $\mathbf{B}$ such that the equation $\mathbf{B}\vec{t} = \vec{0}$:

$$\begin{bmatrix} & & \\ & \mathbf{B} & \\ & & \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

represents the flow conservation constraints for the network in Figure 2.

Hint: Each row is the constraint of an intersection. You can construct $\mathbf{B}$ using only $0, 1$, and -1 entries. This matrix is called the **incidence matrix**. What constraint does each column of $\mathbf{B}$ represent?

(d) Again, suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. Then, determine the subspace of all valid traffic flows for the network of Figure 2. Notice that the set of all vectors $\vec{t}$ that satisfy $\mathbf{B}\vec{t} = \vec{0}$ is exactly the null space of the matrix $\mathbf{B}$. That is, we can find all valid traffic flows by computing the null space of $\mathbf{B}$. What is the dimension of the nullspace?

(e) Notice that we can represent the Berkeley student's measurement as $\mathbf{M}_B\vec{t}$, where:

$$\mathbf{M}_B\vec{t} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \vec{t} = \begin{bmatrix} t_3 \\ t_5 \end{bmatrix}$$

Write a matrix $\mathbf{M}_S$ that can be used to represent the Stanford student's measurement.

(f) Now let us analyze more general road networks. Say there is a road network graph G, with incidence matrix $\mathbf{B}_G$. If $\mathbf{B}_G$ has a k-dimensional null space, does this mean measuring the flows along ***any*** k **roads** is always sufficient to recover all of the true flows? Prove or give a counterexample.

Hint: Consider the Stanford student from part (b).

(g) (**Practice**) Assume that $\vec{u}$ and $\vec{t}$ are distinct valid flows, that is $\mathbf{B}_G\vec{u} = \mathbf{B}_G\vec{t} = \vec{0}$. Can you recover all of the network's true flows if $(\vec{u} - \vec{t})$ belongs to the nullspace of $\mathbf{M}_S$?

Clarification: A "valid" flow is one that is possible without violating the constraints on the nodes (so flow in must equal to flow out). There may be many valid flows, but only one "true" flow.

(h) (**Challenge: Practice**) If the incidence matrix $\mathbf{B}_G$ has a k-dimensional null space, does this mean we can **always pick a set of** k **roads** such that measuring the flows along these roads is sufficient to recover the exact flows? Prove or give a counterexample.

2. Noisy Images (PRACTICE)

Learning Goal: *The imaging lab uses the eigenvalues of the masking matrix to understand which masks are better than others for image reconstruction in the presence of additive noise. This problem explores the underlying mathematics.*

In lab, we used a single pixel camera to capture many measurements of an image $\vec{i}$. A single scalar measurement s_i is captured using a mask $\vec{h}_i$ such that $s_i = \vec{h}_i^T \vec{i}$. Many measurements can be expressed as a matrix-vector multiplication of the masks with the image, where the masks lie along the rows of the matrix.

$$\begin{bmatrix} s_1 \\ \vdots \\ s_N \end{bmatrix} = \begin{bmatrix} \vec{h}_1^T \\ \vdots \\ \vec{h}_N^T \end{bmatrix} \vec{i} \tag{1}$$

$$\vec{s} = \mathbf{H}\vec{i} \tag{2}$$

In the real world, noise, $\vec{w}$, creeps into our measurements, so instead,

$$\vec{s} = \mathbf{H}\vec{i} + \vec{w} \tag{3}$$

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(a) Express $\vec{i}$ in terms of $\mathbf{H}$ (or its inverse), $\vec{s}$, and $\vec{w}$. Assume $\mathbf{H}$ is invertible. (*Hint:* Think about what you did in the imaging lab.)

(b) Depending on how large or small the eigenvalues of $\mathbf{H}^{-1}$ are, we will amplify or attenuate our measurement's noise. The eigenvalues of $\mathbf{H}^{-1}$ are actually related to the eigenvalues of $\mathbf{H}$! Show that if λ is an eigenvalue of a matrix $\mathbf{H}$, then $\frac{1}{\lambda}$ is an eigenvalue of the matrix $\mathbf{H}^{-1}$.
Hint: Start with an eigenvalue λ and one corresponding eigenvector $\vec{v}$, such that they satisfy $\mathbf{H}\vec{v} = \lambda\vec{v}$.

(c) We are going to try different $\mathbf{H}$ matrices in this problem and compare how they deal with noise. Run all of the cells in the attached IPython notebook. Which matrix performs best in reconstructing the original image and why? What do you observe regarding the eigenvalues of matrices $\mathbf{H_1}, \mathbf{H_2}$ and $\mathbf{H_3}$? What special matrix is $\mathbf{H}_1$? Notice that each plot in the iPython notebook returns the result of trying to image a noisy image as well as the minimum absolute value of the eigenvalue of each matrix. Comment on the effect of small eigenvalues on the noise in the image.

(d) Now, because there is noise in our measurements, there will be noise in our recovered image. However, the noise is scaled. The noise in the recovered image, $\hat{\vec{w}}$, is related to $\vec{w}$, but it is transformed by $\mathbf{H}^{-1}$. Specifically,

$$\hat{\vec{w}} = \mathbf{H}^{-1}\vec{w} \tag{4}$$

To analyze how this transformation alters $\vec{w}$, consider representing $\vec{w}$ as a linear combination of the eigenvectors of $\mathbf{H}^{-1}$,

$$\vec{w} = \alpha_1\vec{b}_1 + \ldots + \alpha_N\vec{b}_N. \tag{5}$$

Where, $\vec{b}_i$ is $\mathbf{H}^{-1}$'s eigenvector corresponding to eigenvalue $\frac{1}{\lambda_i}$.

Show that we can express the recovered image's noise as,

$$\hat{\vec{w}} = \mathbf{H}^{-1}\vec{w} = \alpha_1\frac{1}{\lambda_1}\vec{b}_1 + \ldots + \alpha_N\frac{1}{\lambda_N}\vec{b}_N \tag{6}$$

Depending on the size of the eigenvalues, noise in the recovered image will be amplified or attenuated. For eigenvectors with large eigenvalues, will the noise signal along those eigenvectors be amplified of attenuated? For eigenvectors with small eigenvalues, will the noise signal along those eigenvectors be amplified of attenuated?

3. The Dynamics of Romeo and Juliet's Love Affair (PRACTICE)

In this problem, we will study a discrete-time model of the dynamics of Romeo and Juliet's love affair—adapted from Steven H. Strogatz's original paper, *Love Affairs and Differential Equations*, Mathematics Magazine, 61(1), p.35, 1988, which describes a continuous-time model.

Let $R[n]$ denote Romeo's feelings about Juliet on day n, and let $J[n]$ quantify Juliet's feelings about Romeo on day n. If $R[n] > 0$, it means that Romeo loves Juliet and inclines toward her, whereas if $R[n] < 0$, it means that Romeo is resentful of her and inclines away from her. A similar interpretation holds for $J[n]$, which represents Juliet's feelings about Romeo.

A larger $|R[n]|$ represents a more intense feeling of love (if $R[n] > 0$) or resentment (if $R[n] < 0$). If $R[n] = 0$, it means that Romeo has neutral feelings toward Juliet on day n. Similar interpretations hold for larger $|J[n]|$ and the case of $J[n] = 0$.

We model the dynamics of Romeo and Juliet's relationship using the following coupled system of linear evolutionary equations:

$$R[n+1] = aR[n] + bJ[n], \quad n = 0, 1, 2, \dots$$

and

$$J[n+1] = cR[n] + dJ[n], \quad n = 0, 1, 2, \dots,$$

which we can rewrite as

$$\vec{s}[n+1] = \mathbf{A}\,\vec{s}[n],$$

where

$$\vec{s}[n] = \begin{bmatrix} R[n] \\ J[n] \end{bmatrix}$$

denotes the state vector and

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the state transition matrix for our dynamic system model.

The parameters a and d capture the linear fashion in which Romeo and Juliet respond to their own feelings, respectively, about the other person. It's reasonable to assume that $a, d > 0$, to avoid scenarios of fluctuating day-to-day mood swings. Within this positive range, if $0 < a < 1$, then the effect of Romeo's own feelings about Juliet tend to fizzle away with time (in the absence of influence from Juliet to the contrary), whereas if $a > 1$, Romeo's feelings about Juliet intensify with time (in the absence of influence from Juliet to the contrary). A similar interpretation holds when $0 < d < 1$ and $d > 1$.

The parameters b and c capture the linear fashion in which the other person's feelings influence $R[n]$ and $J[n]$, respectively. These parameters may or may not be positive. If $b > 0$, it means that the more Juliet shows affection for Romeo, the more he loves her and inclines toward her. If $b < 0$, it means that the more Juliet shows affection for Romeo, the more resentful he feels and the more he inclines away from her. A similar interpretation holds for the parameter c.

All in all, each of Romeo and Juliet has four romantic styles, which makes for a combined total of sixteen possible dynamic scenarios. The fate of their interactions depends on the romantic style each of them exhibits, the initial state, and the values of the entries in the state transition matrix $\mathbf{A}$. In this problem, we'll explore a subset of the possibilities.

(a) Consider the case where $a + b = c + d$ in the state-transition matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

i. Show that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_1. Also determine the other eigenpair $(\lambda_2, \vec{v}_2)$. Your expressions for λ_1, λ_2, and $\vec{v}_2$ must be in terms of one or more of the parameters a, b, c, and d.

ii. Consider the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

i. Determine the eigenpairs for this system.

ii. Determine all the *fixed points* of the system. That is, find the set of points such that if Romeo and Juliet start at, or enter, any of those points, they'll stay in place forever: $\{\vec{s}_* \mid \mathbf{A}\vec{s}_* = \vec{s}_*\}$. Show these points on a diagram where the x and y-axes are $R[n]$ and $J[n]$.

iii. Determine representative points along the state trajectory $\vec{s}[n]$, $n = 0, 1, 2, \ldots$, if Romeo and Juliet start from the initial state

$$\vec{s}[0] = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

iv. Suppose the initial state is $\vec{s}[0] = \begin{bmatrix} 3 & 5 \end{bmatrix}^T$. Determine a reasonably simple expression for the state vector $\vec{s}[n]$. Find the limiting state vector

$$\lim_{n \to \infty} \vec{s}[n].$$

(b) Consider the setup in which

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

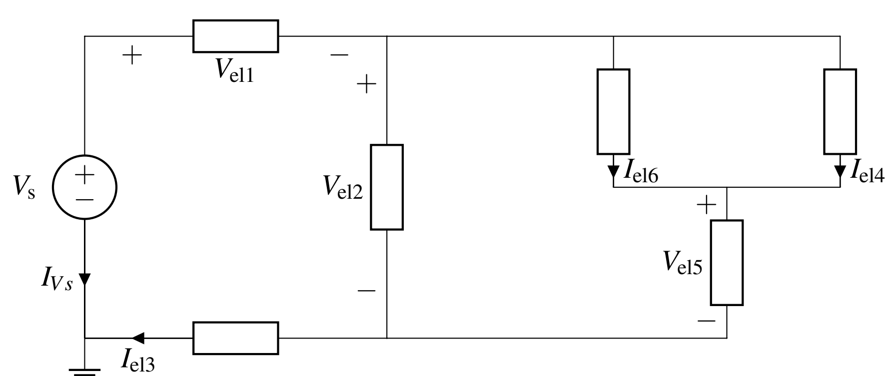
In this scenario, if Juliet shows affection toward Romeo, Romeo's love for her increases, and he inclines toward her. The more intensely Romeo inclines toward her, the more Juliet distances herself. The more Juliet withdraws, the more Romeo is discouraged and retreats into his cave. But the more Romeo inclines away, the more Juliet finds him attractive and the more intensely she conveys her affection toward him. Juliet's increasing warmth increases Romeo's interest in her, which prompts him to incline toward her—again!

Predict the outcome of this scenario before you write down a single equation.

Then determine a complete solution $\vec{s}[n]$ in the simplest of terms, assuming an initial state given by $\vec{s}[0] = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. As part of this, you must determine the eigenvalues and eigenvectors of the $\mathbf{A}$.

Plot (by hand, or otherwise without the assistance of any scientific computing software package), on a two-dimensional plane (called a *phase plane*)—where the horizontal axis denotes $R[n]$ and the vertical axis denotes $J[n]$—representative points along the trajectory of the state vector $\vec{s}[n]$, starting from the initial state given in this part. Describe, in plain words, what Romeo and Juliet are doing in this scenario. In other words, what does their state trajectory look like? Determine $||\vec{s}[n]||^2$ for all $n = 0, 1, 2, \ldots$ to corroborate your description of the state trajectory.

4. Intro to Circuits (MANDATORY - Not in scope for Midterm 1)



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(a) How many nodes does the above circuit have? Label them.
Note: The ground node has been selected for you, so you don't need to label that, but you need to include it in your node count.

(b) Notice that elements 1 - 6 and the voltage source V_s have either the *voltage across* or the *current through* them not labeled. Label the missing *voltages across* or *currents through* for elements 1 - 6, and the voltage source V_s, so that they all follow **passive sign convention**.

(c) Express all element voltages (including the element voltage across the source, V_s) as a function of node voltages. This will be specific to the node labeling you chose in part (a).

(d) Write one KCL equation that involves the currents of elements 1 and 2.
Hint: This will **not** be specific to your node labeling.

(e) Write a KVL equation for all the loops that contain the voltage source V_s. These equations should be a function of element voltages and the voltage source V_s.
Hint: This will also **not** be specific to your node labeling. There are 3 such loops in the circuit.

5. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

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