EECS 16A
Lecture 0B
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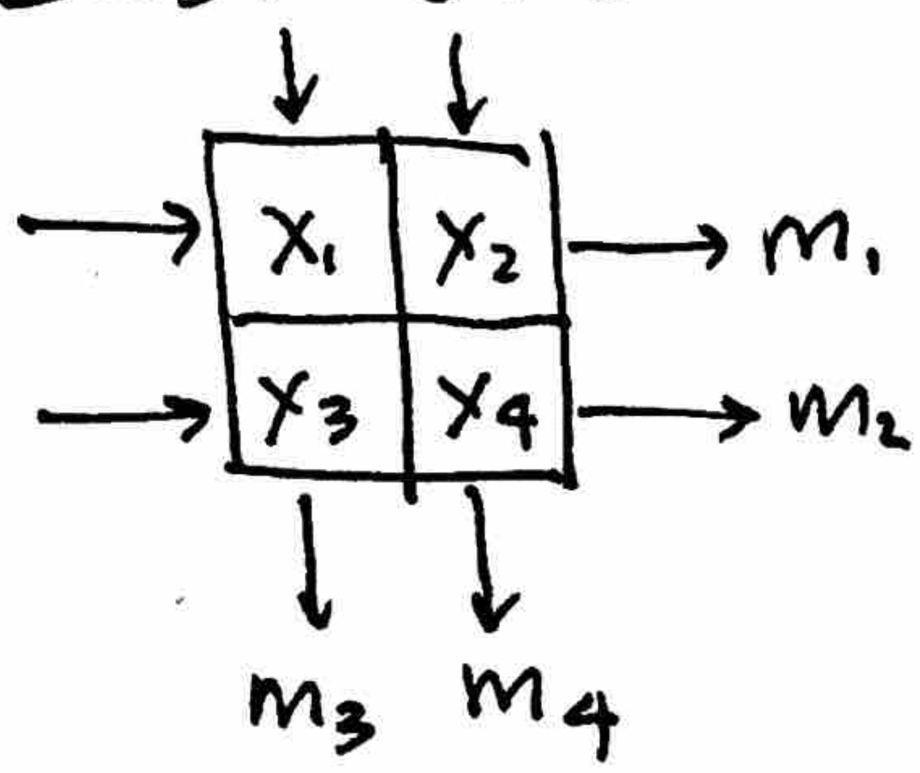
TODAY:
- Systems of linear equations
- Gaussian Elimination

Last Class:



$$\begin{aligned} x_1 + x_2 &= m_1 \\ x_3 + x_4 &= m_2 \\ x_1 + x_3 &= m_3 \\ x_2 + x_4 &= m_4 \end{aligned}$$

System of linear Equations

collection of Equations

$f(x) = b$ constant

① $f(\alpha x) = \alpha f(x)$
② $f(x+y) = f(x) + f(y)$

properties of a linear equation for 1 variable

ex) $f(x) = x^2$

① $f(\alpha x) = (\alpha x)^2 = \alpha^2 x^2$
$\alpha f(x) = \alpha x^2$ not equal!

ex) $f(x) = a^2 x$ a is a constant

const. unknown

$f(x_1, x_2, \dots x_n) = b$

① $f(\alpha x_1, \alpha x_2, \dots \alpha x_n) = \alpha f(x_1, x_2, \dots, x_n)$

② $f(x_1+y_1, x_2+y_2, \dots, x_n+y_n) = f(x_1\ x_2 \dots x_n) + f(y_1\ y_2 \dots y_n)$

$$\text{meas}_1 = \alpha_1 \cdot (\text{Beach}) + \beta_1 \cdot (\text{person})$$
$$\text{meas}_2 = \alpha_2 \cdot (\text{Beach}) + \beta_2 \cdot (\text{person})$$

known   unknown

SIMPLIFY:

$$\tfrac{1}{2}x + \tfrac{1}{2}y = 4 \quad \text{E1}$$
$$\tfrac{1}{4}x + \tfrac{3}{4}y = 5 \quad \text{E2}$$

$\Rightarrow$

<u>Solve systematically</u>

① Make coeff x in E1 $= 1$
"Normalizing E1"

$$\begin{cases} x + y = 8 & \leftarrow \\ \tfrac{1}{4}x + \tfrac{3}{4}y = 5 \end{cases}$$

② Use new E1 to eliminate x from E2

E2 $- \tfrac{1}{4}$ E1

$$\begin{cases} x + y = 8 \\ \tfrac{1}{2}y = 3 \end{cases}$$

③ Solve for y

$$\begin{cases} x + y = 8 \\ y = 6 \end{cases}$$

④ Backsubstitution

E1 $-$ E2

$$\begin{cases} x = 2 \\ y = 6 \end{cases}$$

Augmented Matrix

$$\left[\begin{array}{cc|c} \tfrac{1}{2} & \tfrac{1}{2} & 4 \\ \tfrac{1}{4} & \tfrac{3}{4} & 5 \end{array}\right] \begin{array}{l} \leftarrow R1 \\ \leftarrow R2 \end{array}$$

$R1 \leftarrow R1 \times 2$

$$\left[\begin{array}{cc|c} 1 & 1 & 8 \\ \tfrac{1}{4} & \tfrac{3}{4} & 5 \end{array}\right]$$

$R2 \leftarrow R2 - \tfrac{1}{4} R1$

$$\left[\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & \tfrac{1}{2} & 3 \end{array}\right]$$

$R2 \leftarrow 2 \times R2$

$$\left[\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 6 \end{array}\right]$$

$R1 \leftarrow R1 - R2$

$$\begin{array}{c} \left[\begin{array}{cc|c} 1 & 0 & \cancel{8}\,2 \\ 0 & 1 & 6 \end{array}\right] \\ \begin{array}{cc} \uparrow x & \uparrow y \end{array} \end{array}$$

$x = \cancel{8}\,2$
$y = 6$

$$\begin{aligned}\alpha_1 x + \beta_1 y &= m_1 \\ \alpha_2 x + \beta_2 y &= m_2\end{aligned} \iff \left[\begin{array}{cc|c} \alpha_1 & \beta_1 & m_1 \\ \alpha_2 & \beta_2 & m_2 \end{array}\right] \iff \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$$

"Matrix-Vector Form"

Example 2

$$\begin{aligned}2x + 3y &= 8 \\ 2x + 3y &= 6\end{aligned} \rightarrow \left[\begin{array}{cc|c} 2 & 3 & 8 \\ 2 & 3 & 6 \end{array}\right]$$

$R1 \leftarrow R1/2$

$$\left[\begin{array}{cc|c} 1 & 3/2 & 4 \\ 2 & 3 & 6 \end{array}\right]$$

$R2 \leftarrow R2 - 2R1$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 3/2 & 4 \\ 0 & 0 & -2 \end{array}\right]$$

$\boxed{\text{NO SOLUTION}}$ $\qquad 0 = -2$ UH. OH.

Example 3

$$\begin{aligned}x + 4y &= 6 \\ 2x + 8y &= 12\end{aligned} \rightarrow \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & 8 & 12 \end{array}\right]$$

Normalize ✓

$R2 \leftarrow R2 - 2R1$

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

$0 = 0$

$x + 4y = 6$ $\qquad$ x is basic variable

choose y → solve x

y is <u>free variable</u>

$\boxed{\text{Infinite solutions}}$

GAUSSIAN ELIMINATION

n variables, m equations

$$
\begin{matrix}
\alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n = \beta_1 \\
\alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n = \beta_2 \\
\vdots \qquad\qquad\qquad\qquad \vdots \\
\alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots \alpha_{mn}x_n = \beta_m
\end{matrix}
\Rightarrow
\left[\begin{array}{cccc|c}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} & \beta_1 \\
\alpha_{21} & & & \vdots & \beta_2 \\
\vdots & & & \vdots & \vdots \\
\alpha_{m1} & & \cdots & \alpha_{mn} & \beta_m
\end{array}\right]
$$

What operations did we do on each row?

- Rescaling (× by nonzero constant)
- Add a scalar multiple of a row to another row
- Swap rows

First nonzero entry of each row $\boxed{\text{DEF}}$ "leading entry" of that row.

$$
\begin{array}{c}
\rightarrow \\
\rightarrow
\end{array}
\left[\begin{array}{cc|c}
1 & 2 & 4 \\
0 & 3 & 5
\end{array}\right]
$$

For every row i (start with $i=1$):

① Swap row i with a row below to leading entry as far left as possible

$$
\begin{array}{c}
\rightarrow \\
\rightarrow
\end{array}
\left[\begin{array}{cc|c}
10 & 2 & 5 \\
3 & 4 & 6
\end{array}\right]
$$

② scale row i so leading entry is 1

③ For all rows below row i ($j = i+1 \ldots m$)
↳ use the ith row to cancel x_i from row j

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

"Row Echelon Form"

$*$ - any number

- leading entries are 1
- leading entries are all to the right of prior leading entry
- all zeros below each leading entry

④ Backsubstitution

Going in reverse order throug the rows (row $i = m \ldots 1$)

Cancel the entries above the leading entry.

example result

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

"Reduced Row Echelon Form"

"rref"

Variables corresponding to columns with leading entries are called <u>basic Variables</u>.

other variables are called <u>free variables</u>.

$$x + y = 2$$
$$x - y = 1$$
$$2x - 2y = 2$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & -2 & 2 \end{array}\right]$$

$R2 \leftarrow R2 - R1$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -1 \\ 2 & -2 & 2 \end{array}\right]$$

$R3 \leftarrow R3 - 2R1$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & -4 & -2 \end{array}\right]$$

$R2 \leftarrow R2/-2$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1/2 \\ 0 & -4 & -2 \end{array}\right]$$

$R3 \leftarrow \cancel{-4R2}$ $R3 + 4R2$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right]$$

$R1 \leftarrow R1 - R2$

$$\left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right]$$

DONE
w/ GE

$x = 3/2 \quad y = 1/2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

UNIQUE SOLUTION

$$\left[\begin{array}{cccc|c} & & & & \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$0 = 1 \qquad 0 = 5$

NO SOLUTIONS

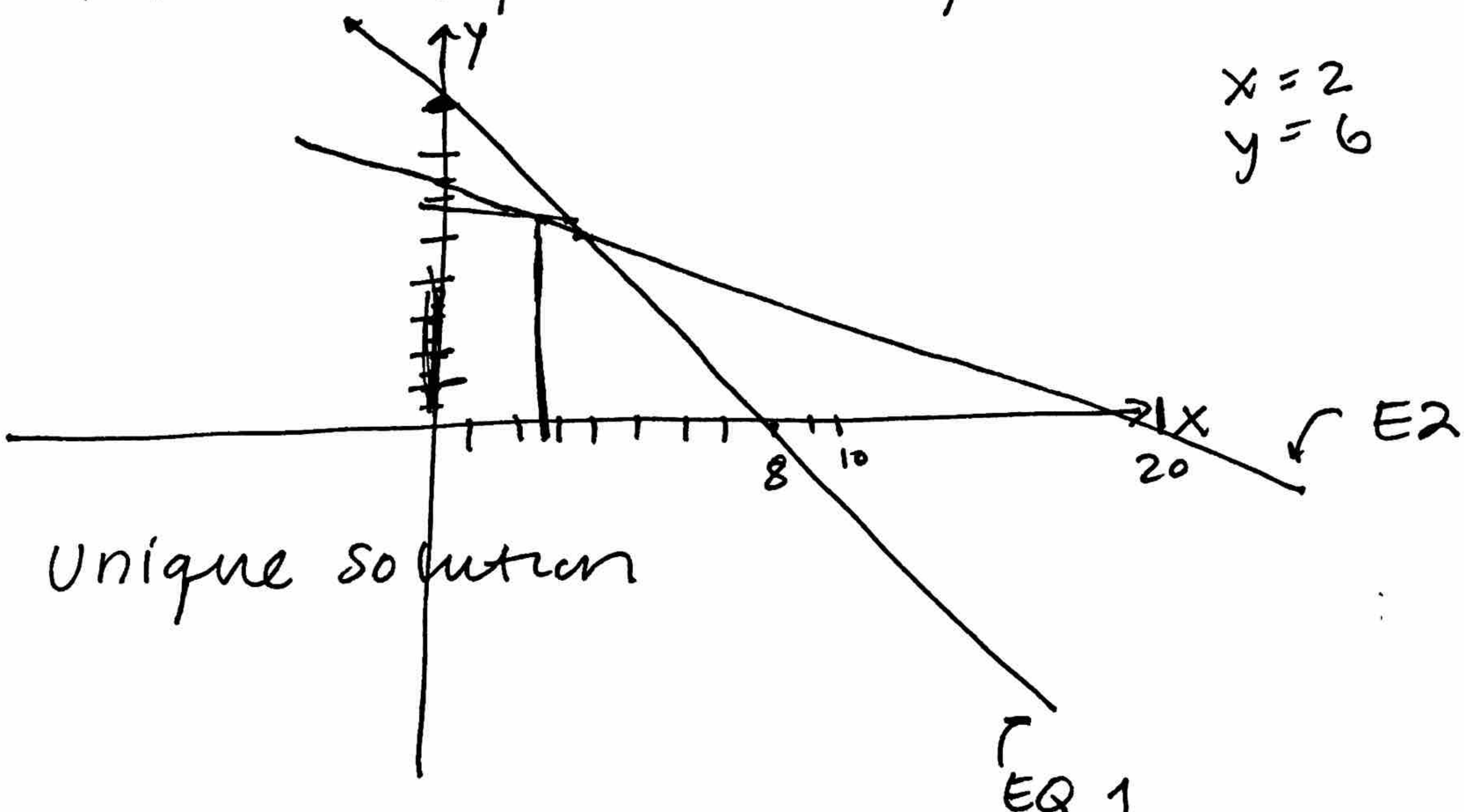
$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Free variable

$\frac{1}{2}x + \frac{1}{2}y = 4 \Rightarrow y = -x + 8$ (E1)

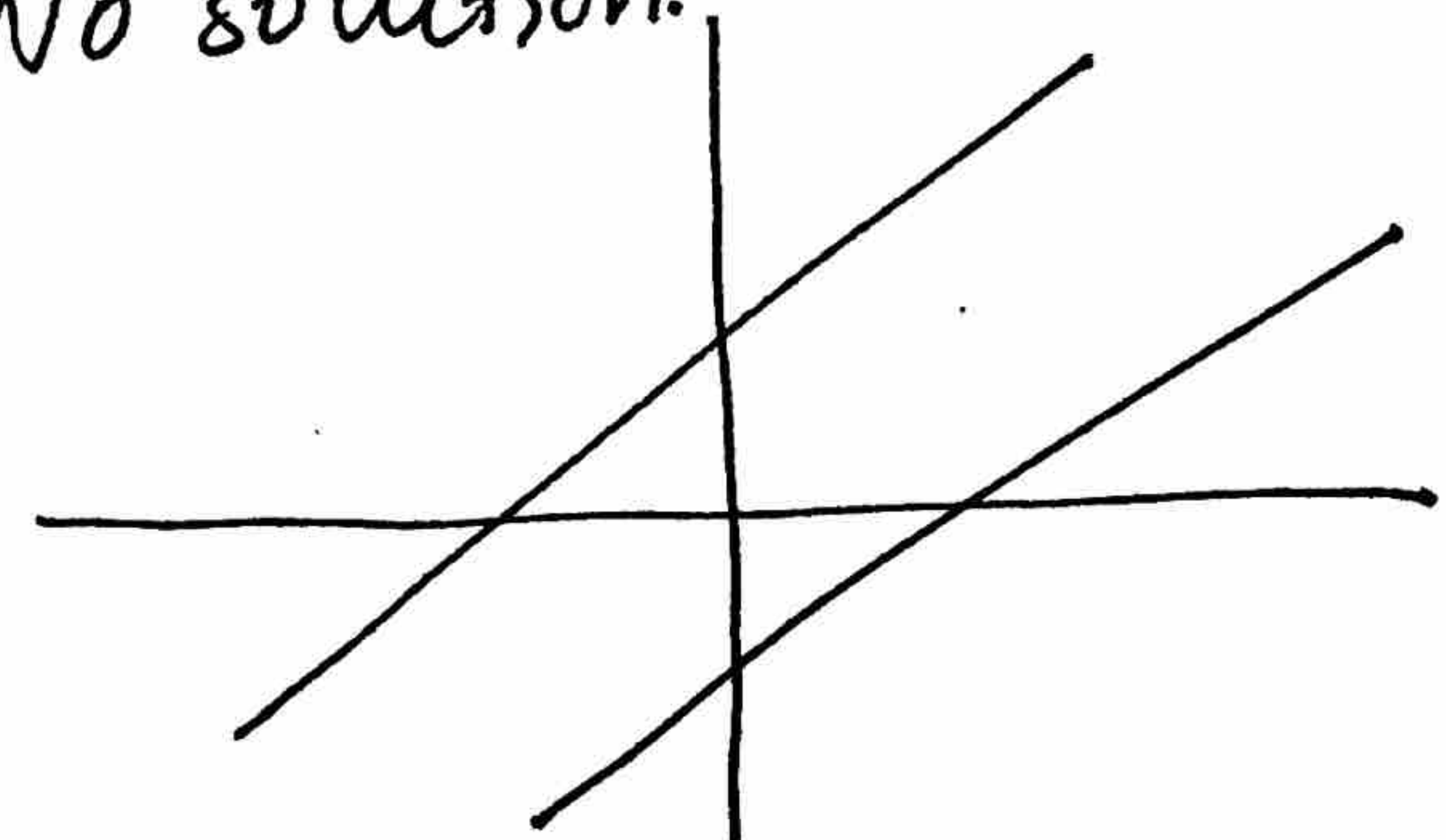
$\frac{1}{4}x + \frac{3}{4}y = 5 \Rightarrow y = -\frac{1}{3}x + 6\frac{1}{3}$ (E2)

$x = 2$
$y = 6$



Unique solution

No solution:



∞ solutions: