EECS 16A
Lecture 0C
June 24, 2020
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Today:
- More practice with Gaussian Elimination
- Vectors + Matricies

Reminder: HW0A due tonight!
Lab starts today!

Yesterday:

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

* any number

Result of Gaussian Elimination

"Row Reduced Echelon Form"
"rref"

leading entries
"pivots"

Variables associated with columns containing leading entries are <u>basic variables</u>.

Other variables are <u>free variables</u>.

Example

$$\begin{aligned} 2y + 3z &= 2 \\ x + y \quad &= 1 \end{aligned}$$

① Go through rows:
- → move leading entries to the left (swap)
- → set leading entries to 1 (rescale)
- → clear everything below leading entries

② Backsubstitution
- → clear everything above leading entries

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 2 & 3 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

swap $R_1 R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & 2 \end{array}\right]$$

$R_2 \leftarrow R_2/2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 3/2 & 1 \end{array}\right]$$

$R1 \leftarrow R1 - R2$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -3/2 & 0 \\ 0 & 1 & 3/2 & 1 \end{array}\right] \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

$$x - 3/2\, z = 0$$
$$y + 3/2\, z = 1$$

$x = \frac{3}{2} z$

$y = 1 - \frac{3}{2} z$

$z =$ free

choose what you want

$\boxed{\text{Example}}$ 3 eqs. 2 unknowns

$$
\begin{aligned}
x + y &= 2 \\
2x - y &= 1 \\
4x + 3y &= 4
\end{aligned}
\rightarrow
\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 4 & 3 & 4 \end{array}\right]
$$

$$
\xrightarrow{R2 \leftarrow R2 - 2R1}
\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 4 & 3 & 4 \end{array}\right]
\xrightarrow{R3 \leftarrow R3 - 4R1}
\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & -1 & -4 \end{array}\right]
$$

$$
\xrightarrow{R2 \leftarrow R2/-3}
\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -4 \end{array}\right]
\xrightarrow{R3 \leftarrow R3 + R2}
\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{array}\right]
$$

$0 = -3$

No solution

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 4 & 3 \end{bmatrix}}_{\text{Matrix}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{vectors}} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

"Matrix - Vector Form"

Matrix

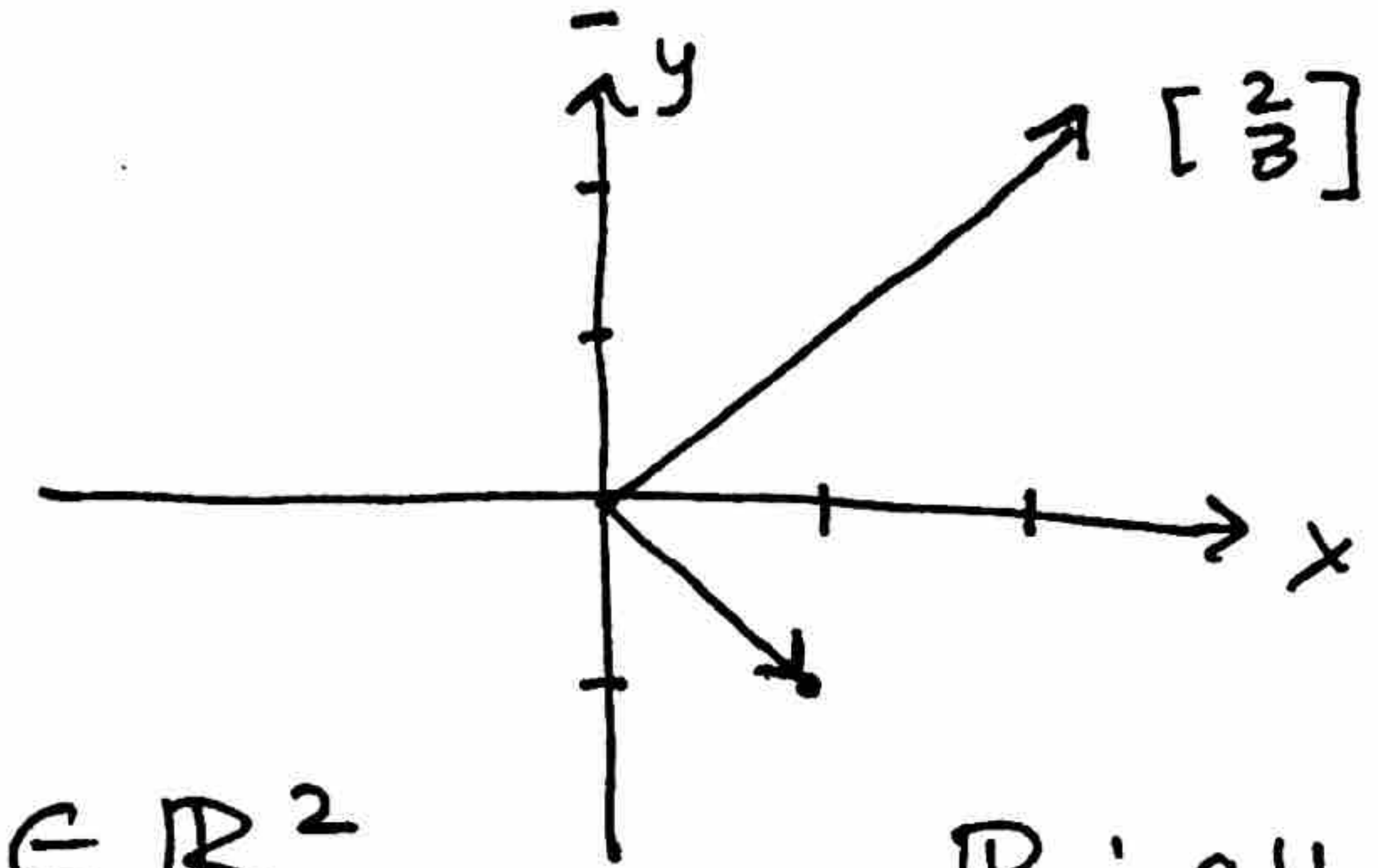
vectors

Definitions

A <u>vector</u> is an ordered list of numbers.

ex) $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ← x, ← y

arrow on top ⇒ vector



$\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}^2$

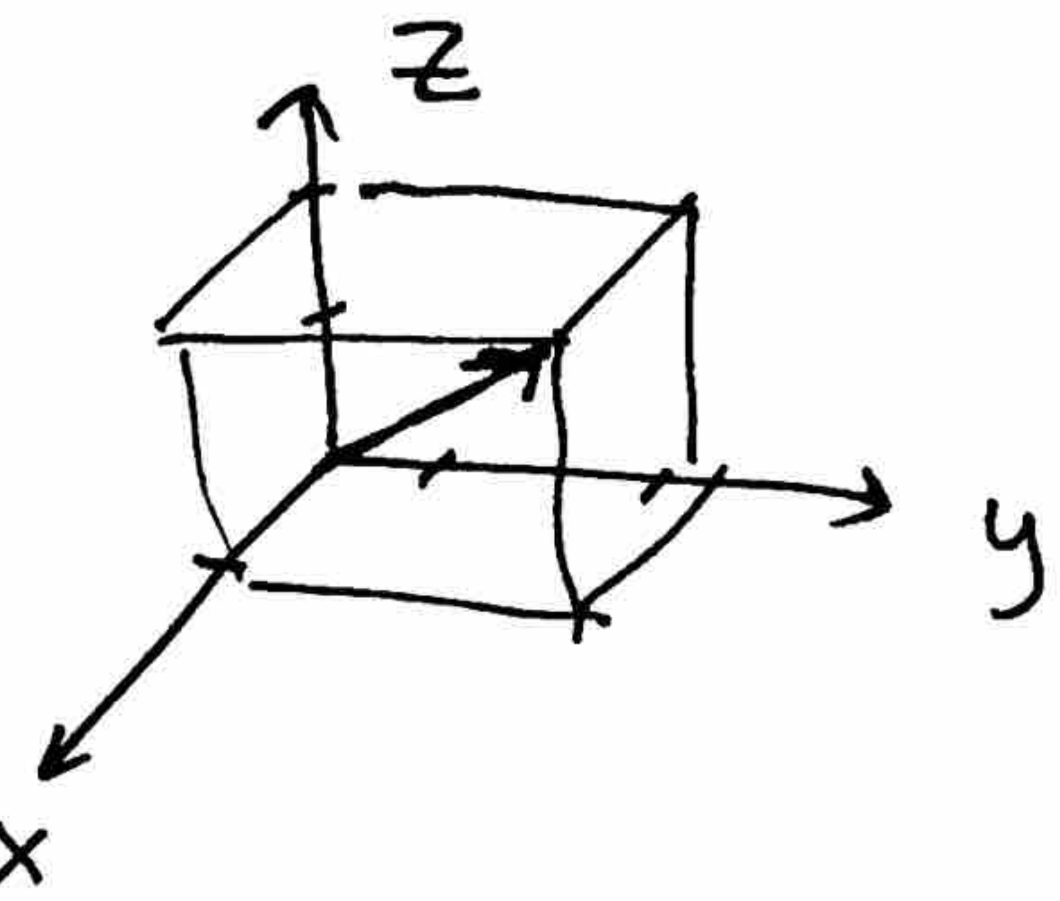
$\mathbb{R}^2$: all pairs of real numbers $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

$\mathbb{R}$: all real numbers

$5 \in \mathbb{R}$

$a \in \mathbb{R}$

ex) $\begin{array}{l} x \\ y \\ z \end{array} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \in \mathbb{R}^3$



ex) $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{x} \in \mathbb{R}^n$

Vector Operations

Vector Addition

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_2 \\ x_2 + y_2 \end{bmatrix}$$

ex) $$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

Multiplication between scalar and vector:

$$\alpha \vec{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} \qquad \text{ex) } 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$\alpha \in \mathbb{R}$

Transpose: flips orientation of vector

$$\underbrace{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{column vector}} \qquad \vec{x}^T = \underbrace{\begin{bmatrix} x_1 & x_2 \end{bmatrix}}_{\text{row vector}}$$

By convention, we'll assume all vectors are column vectors

ex) $$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

col. vec

$$\underbrace{\left(\vec{x}^T\right)^T}_{\cancel{\text{col vec}}\ \text{row}} = \underbrace{\vec{x}}_{\text{col vec.}}$$

A <u>matrix</u> is a grid of numbers.
(array)

ex)

m rows

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

n columns

$$A \in \mathbb{R}^{m \times n}$$

size:

$m \times n$

row column

a_{21}

2nd row 1st column

Addition:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+1 \\ 3+1 & 4+0 \end{bmatrix}$$

Multiplication by a scalar

$$\alpha A = \begin{bmatrix} \alpha & 2\alpha \\ 3\alpha & 4\alpha \end{bmatrix}$$

Transpose: $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

each row of the matrix becomes a colum

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T}_{2 \times 3} = \underbrace{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}}_{3 \times 2}$$

Matrix-vector multiplication

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A\vec{x} = \underset{\substack{3\times 3 \\ M \times N}}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \underset{\substack{3\times 1 \\ N}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\substack{3\times 1 \\ M}}{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}}$$

ex) $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ 4x \;\; 5y + 6z \end{bmatrix} \updownarrow 2$$

ex) $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 10 \\ 100 \end{bmatrix} = \begin{bmatrix} 1 + 20 + 300 \\ 4 + 50 + 600 \end{bmatrix} = \begin{bmatrix} 321 \\ 654 \end{bmatrix}$$

ex) $$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Identity Matrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1x_1 + 0x_2 + 0x_3 + 0x_4 \\ 0x_1 + 1x_2 + 0x_3 + 0x_4 \\ \vdots \\ \end{bmatrix}$$

I square matrix with ones along the diagonal

"Row perspective on matrix-vector multiplication"

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \qquad \vec{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \qquad \vec{a}_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$$

$$= \begin{bmatrix} x_1 a_{11} \\ x_1 a_{21} \\ x_1 a_{31} \end{bmatrix} + \begin{bmatrix} x_2 a_{12} \\ x_2 a_{22} \\ x_2 a_{32} \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} x_1 a_{11} + x_2 a_{12} + x_3 a_{13} \\ x_1 a_{21} + \quad \cdots \quad x_3 a_{23} \\ x_1 a_{31} + \quad \cdots + x_3 a_{33} \end{bmatrix}$$

"Column perspective on matrix-vector multiplication"