EEC S 16A
Lecture 0D
June 25, 2020
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TODAY:
- span
- linear dependence
- proofs

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A \in \mathbb{R}^{m \times n}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x} \in \mathbb{R}^n} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b} \in \mathbb{R}^m}$$

$\vec{a}_1$ ... $\vec{a}_n$

m equations
n unknowns

$$\boxed{A\vec{x} = \vec{b}}$$

coeff — unknowns — measurements

$$\boxed{x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b}}$$

Column view

- Does $A\vec{x} = \vec{b}$ have a solution?

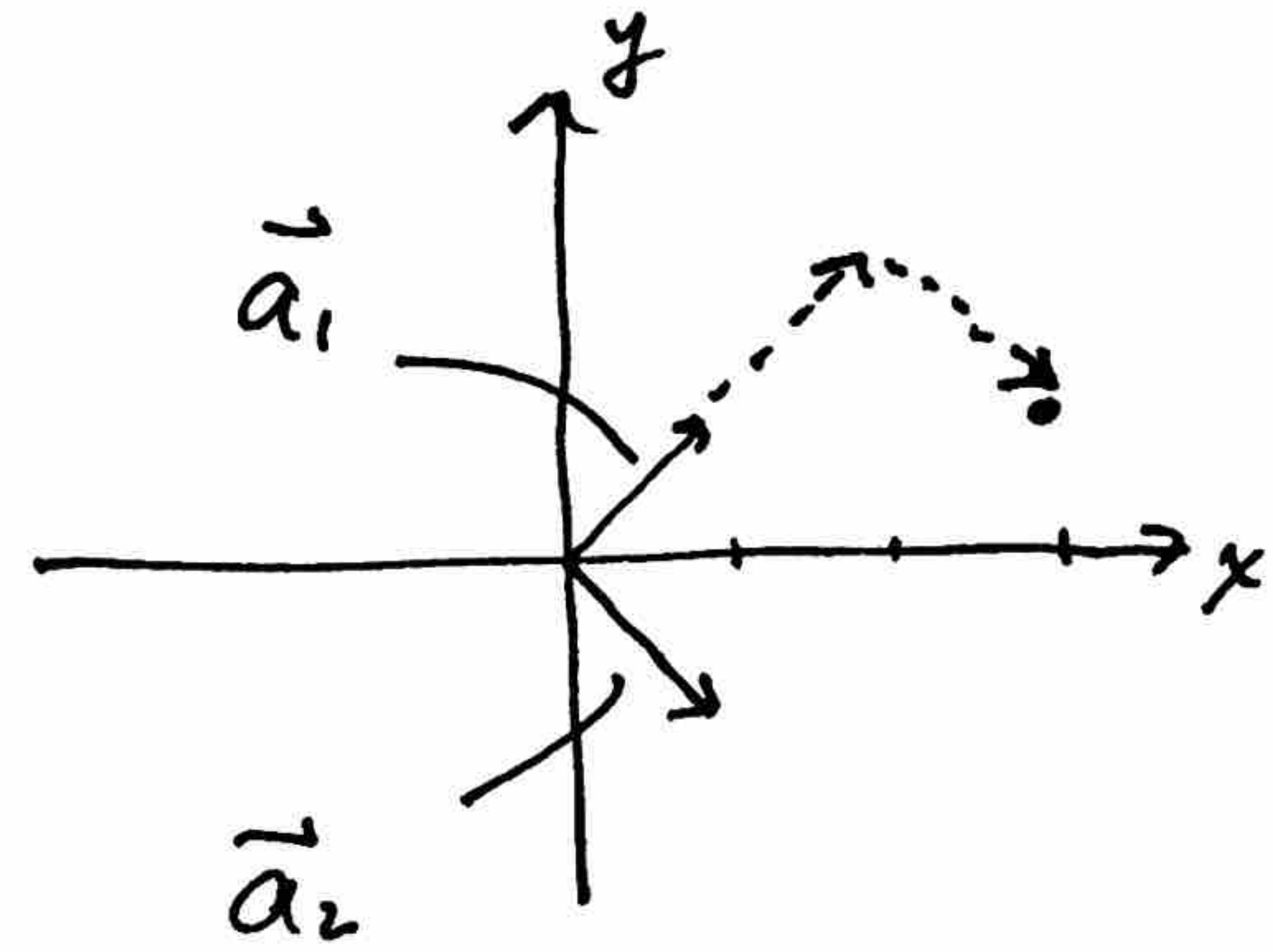
$\Updownarrow$

- Can we express $\vec{b}$ as a linear combination of the columns of A?

Span

If we have vectors $\vec{a}_1 \dots \vec{a}_n$ (these are all the same size)
then $\text{span}\{\vec{a}_1 \dots \vec{a}_n\}$ is the set of all vector $\vec{v}$
that can be written as a linear combination
of $\{\vec{a}_1 \dots \vec{a}_n\}$.

ex) $\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



Is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in the $\text{span}\{\vec{a}_1, \vec{a}_2\}$?

can we find b_1, b_2 such that

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \underline{b_1}\,\vec{a}_1 + \underline{b_2}\,\vec{a}_2 \,?$$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 2\,\vec{a}_1 + 1\,\vec{a}_2$$

Equivalent to:

- set of all vectors $\vec{b}$ such that $\underset{[\vec{a}_1 \dots \vec{a}_n]}{A}\vec{x} = \vec{b}$ has a solution
- $\text{span}\{\vec{a}_1 \dots \vec{a}_n\} = \left\{ \sum_{i=1}^{n} c_i \vec{a}_i \;\middle|\; c_i \in \mathbb{R} \right\}$

all c_i's are real scalars

$c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n$

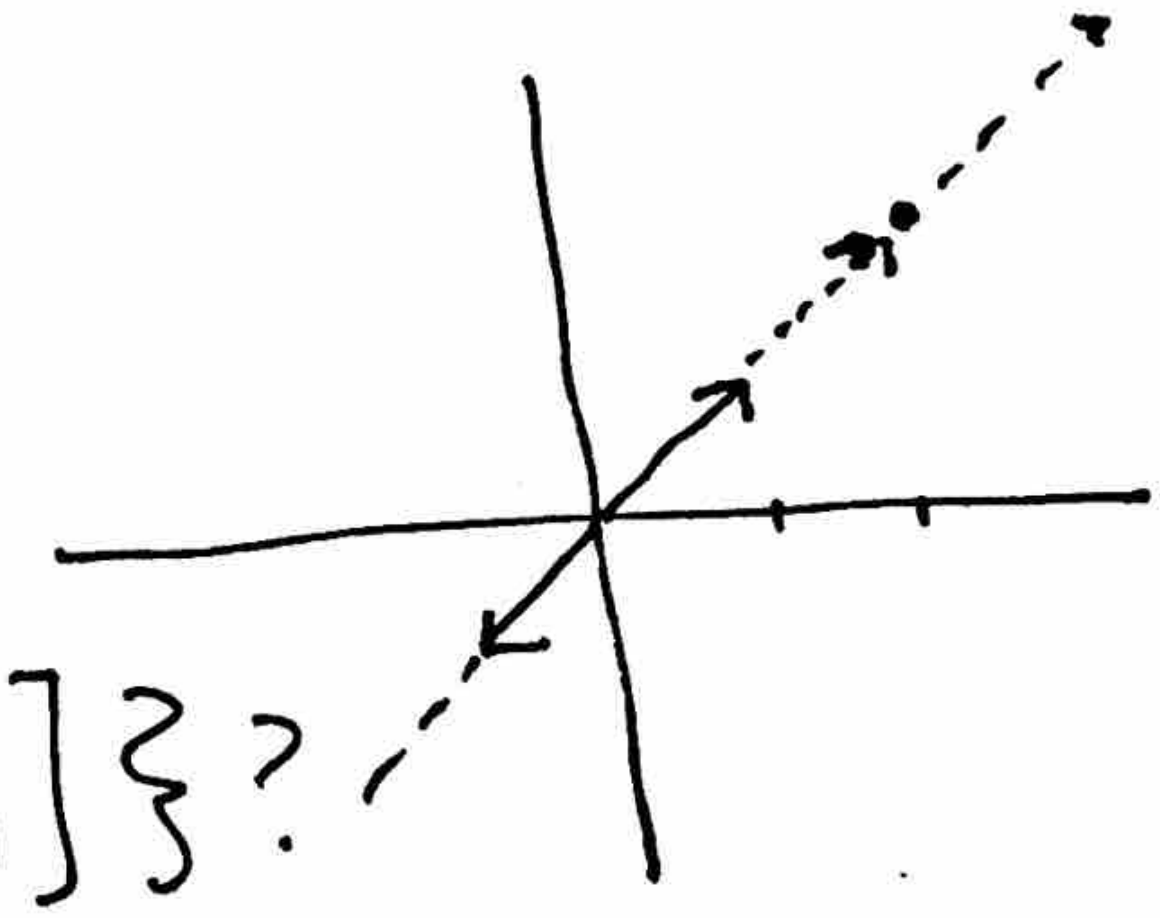
$$\text{span}\{\vec{a}_1 \dots \vec{a}_n\} = \text{span}(A) = \text{range}(A)$$
$$= \text{Columnspace}(A)$$

some more jargon

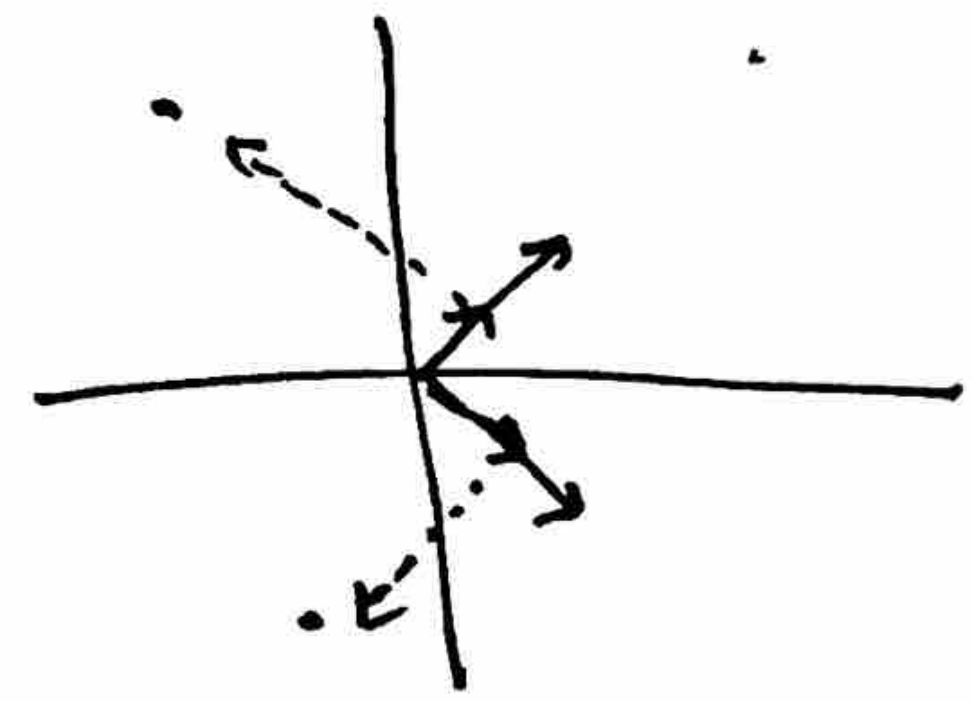
ex) $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}\right\}$

$\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}\vec{x} = \vec{b}$

Is $\begin{bmatrix}2\\2\end{bmatrix}$ in $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}-1\\-1\end{bmatrix}\right\}$?



$= \left\{\begin{bmatrix}\alpha\\\alpha\end{bmatrix} \,\middle|\, \alpha \in \mathbb{R}\right\}$



ex) $\underbrace{\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}}_{\text{collection of vectors}} = \underbrace{\mathbb{R}^2}_{\text{collection of vectors}}$

ASIDE

Students who watch lecture

Students on the class roster

① Check ppl who watched lecture
→ are they on the roster?

② Check ppl on roster
→ did they watch lecture?

$\vec{v} \in \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}1\\-1\end{bmatrix}\right\}$

$\vec{v} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}c_1 + c_2\\c_1 - c_2\end{bmatrix}$ is in $\mathbb{R}^2$ ✓

↖ two elements that are real

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ Is $\vec{u}$ in span$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$?

$$\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & u_1 \\ 1 & -1 & u_2 \end{array}\right]$$

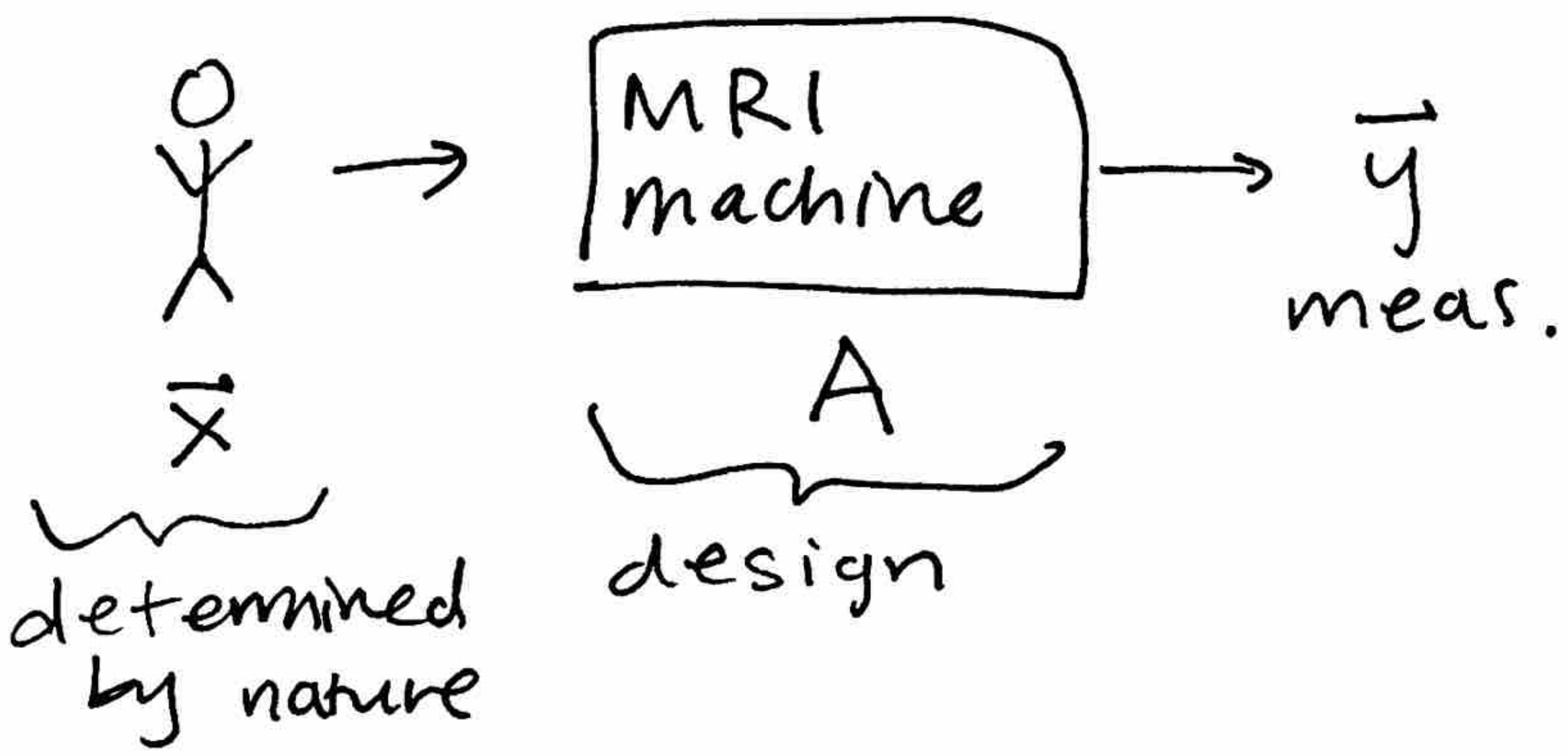
left an exercise to the reader

$$\alpha = \frac{u_1 + u_2}{2} \qquad \beta = \frac{u_1 - u_2}{2}$$

Yes $\vec{u}$ is in the span of $\vec{a_1}$ $\vec{a_2}$!

Proven span$\{\vec{a_1}, \vec{a_2}\} = \mathbb{R}^2$

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\} \stackrel{?}{=} \mathbb{R}^2$$



Linear Dependence

easier to use in proofs

1. A set of vectors $\{\vec{V}_1, \vec{V}_2 \dots \vec{V}_n\}$ are said to be linearly dependent if there exists scalars $\alpha_1, \alpha_2, \dots \alpha_n$, not all zero, such that

$$\sum_{i=1}^{n} \alpha_i \vec{V}_i = \vec{O} = \alpha_1 \vec{V}_1 + \alpha_2 \vec{V}_2 + \dots + \alpha_n \vec{V}_n$$

easier to think about

2. A set of vectors $\{\vec{V}_1, \vec{V}_2 \dots \vec{V}_n\}$ are said to be linearly dependent if one of the vectors can be written as a linear combination of the others.

→ theres some index j

$$\vec{V}_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i \vec{V}_i$$

ex) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ → lin. dep.? Yes!

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ → lin dep? NO!

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ → lin. dep.? Yes!

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ → lin dep? Yes!

THM: Def ① and Def ② are equivalent.

$\vec{V}_1 \ldots \vec{V}_n$ are L.D. by def ① $\Rightarrow$ also L.D by def ②

" " " def ② $\Rightarrow$ " " ①

Def ① $\Rightarrow$ Def ②

$$\sum_{i=1}^{n} \alpha_i \vec{V}_i = \vec{0}$$

where not all α_i are zero

$$\alpha_1 \vec{V}_1 + \alpha_2 \vec{V}_2 + \alpha_3 \vec{V}_3 = \vec{0}$$

at least one of $\alpha_1 \ldots \alpha_3 \neq 0$

α_j is non zero

$$-\alpha_j \vec{V}_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i \vec{V}_i$$

$$\vec{V}_j = \sum_{i=1\, i \neq j}^{n} -\frac{\alpha_i}{\alpha_j} \vec{V}_i$$

$$\vec{V}_j = \sum_{\substack{i=1 \\ j \neq i}}^{n} \beta_i \vec{V}_i$$

$$\beta_i = -\frac{\alpha_i}{\alpha_j}$$

for all $i \neq j$

Def ② $\Rightarrow$ Def ①

$$\vec{V}_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} \beta_i \vec{V}_i$$

$$\vec{0} = \sum_{\substack{i=1 \\ i \neq j}}^{n} \beta_i \vec{V}_i - \vec{V}_j$$

$$\vec{0} = \sum_{i=1}^{n} \beta_i \vec{V}_i \,, \quad \beta_j = -1$$

there are α_i's <u>not all zero</u>

$$\sum_{i=1}^{n} \alpha_i \vec{V}_i = \vec{0}$$

$$\vec{V}_2 = \beta_1 \vec{V}_1 + \beta_3 \vec{V}_3$$

$$\vec{0} = \beta_1 \vec{V}_1 - \vec{V}_2 + \beta_3 \vec{V}_3$$

$\alpha_1 = \beta_1$ $\quad$ $\alpha_2 = -1$ $\quad$ $\alpha_3 = \beta_3$

$$\alpha_1 \vec{V}_1 + \alpha_2 \vec{V}_2 + \alpha_3 \vec{V}_3 = \vec{0}$$

not all α are zero

Tips for doing proofs

1. Write explicity

Beginning & know, assume

End: what you want
to show

→ translate words to math

2. Try a simple example
→ remove complex notation (ex Σ)

3. Work from both ends

4. Use scrap paper

5. You should understand all the
steps!

$\boxed{\text{Thm}}$ $\boxed{\text{If}}$ $A\vec{x}=\vec{b}$ has two or more solutions, $\boxed{\text{then}}$ the columns of A are linearly dependent.

Start: $A\vec{x}=\vec{b}$ has two or more solutions

$$A=[\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_n]$$

$A\vec{x}_1=\vec{b}$ $\quad A\vec{x}_2=\vec{b}$ $\quad$ where $\underline{\underline{\vec{x}_1\neq\vec{x}_2}}$

subtract: $A(\vec{x}_1-\vec{x}_2)=\vec{0}$

$$\vec{\beta}=\vec{x}_1-\vec{x}_2\neq\vec{0}$$

$A\vec{\beta}=\vec{0}$ where at least one $\beta_i\neq 0$

$$[\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_n]\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{bmatrix}=\vec{0}$$

$$\sum_{i=1}^{n}\beta_i\vec{a}_i=\vec{0} \quad =\beta_1\vec{a}_1+\beta_2\vec{a}_2+\dots+\beta_n\vec{a}_n$$

$\cancel{\sum_{i=1}^{n}\alpha_i\vec{a}_i=0}$ where at least one $\overset{\beta_i}{\cancel{\alpha_i}}\neq 0$

End: The columns of A are linearly dependent.

$A\vec{x}_1=\vec{b}$ $\quad A\vec{x}_2=\vec{b}$

$A\vec{x}_1-A\vec{x}_2=\vec{b}-\vec{b}$

$A(\vec{x}_1-\vec{x}_2)=\vec{0}$