EECS 16A
Lecture 1A
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Today:
- another proof on linear dependence
- state transformations
- matrix-matrix multiplication

Allowed Operations

$\vec{x}, \vec{y} \in \mathbb{R}^{\underset{\uparrow}{n}}$ ~~n col.~~ rows

$A, B \in \mathbb{R}^{\underset{\uparrow}{m} \times \underset{\uparrow}{n}}$ m rows n col.

$\alpha, \beta \in \mathbb{R}$

- Add, subtract

ex) $\vec{x} + \vec{y}$ $\quad \cancel{\vec{x} + A}$ Not allowed

$A + B$

- Scalar multiplication

ex) $\alpha \vec{x}$ $\quad \alpha A$

↳ scalar multiplication commutes

$\alpha \vec{x} = \vec{x} \alpha \qquad \alpha A = A \alpha$

$$\alpha \vec{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} = \begin{bmatrix} x_1 \alpha \\ x_2 \alpha \end{bmatrix} = \vec{x} \alpha$$

- Distributive property ↙ vector math ↙ scalar math

ex) $\alpha(\vec{x} + \vec{y}) = \alpha \vec{x} + \alpha \vec{y}$ $\qquad a(x+y) = ax + ay$

$\alpha(A + B) = \alpha A + \alpha B$

$(\alpha + \beta)\vec{x} = \alpha \vec{x} + \beta \vec{x}$

$(A + B)\vec{x} = A\vec{x} + B\vec{x}$ $\qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$

$\cancel{(\vec{x} + \vec{y})A = \underbrace{\vec{x}A}_{\text{not valid}}}$

Not Allowed

$A\vec{x} \neq \underbrace{\vec{x}A}_{\text{Not defined!}}$ Not commutative!

columns of A must equal # of rows of $\vec{x}$ to be defined $A\vec{x}$

Not defined $\cancel{\vec{x}\vec{y}}$

$\boxed{\text{THM}}$ $\boxed{\text{If}}$ the columns of A are linearly dependent $\boxed{\text{then}}$ $A\vec{x} = \vec{b}$ does not have a unique solution.

If p then q

Proof by contradiction
Assume not q
↓
get contradiction
↓
q is true

Start:

columns of A are lin. dep.

$A\vec{x} = \vec{b}$ has unique solution ← assuming not q

→ $\vec{a}_1\ \vec{a}_2 \dots \vec{a}_n$ are columns of A

$$\sum_{i=1}^{n} c_i \vec{a}_i = \vec{0} \Leftrightarrow c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n = \vec{0} \Leftrightarrow \boxed{A\vec{c} = \vec{0}}$$

not all c_i are 0

$\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ $\vec{c} \neq \vec{0}$

→ unique solution $\vec{x}^*$. $A\vec{x}^* = \vec{b}$

Can we get a contradiction?

ex) 0 = 2

ex) contradiction w/ assumptions

⇒ x^* is unique (assumption) : show x^* is not unique. That there are ∞ solution.

$$A\vec{x}^* + A\vec{c} = \vec{b} + \vec{0}$$

$$A(\underbrace{\vec{x}^* + \vec{c}}) = \vec{b}$$

$$\vec{x}^{**} = \vec{x}^* + \vec{c}$$

$\vec{x}^{**}$ is another solution

contradiction!

we know $\vec{x}^{**} \neq \vec{x}^*$ because $\vec{c} \neq \vec{0}$

$$A\vec{c} = \vec{0}$$

$$A(\alpha\vec{c}) = \alpha A\vec{c} = \alpha\vec{0} = \vec{0}$$

$$A\vec{x}^* + A(\alpha\vec{c}) = \vec{b}$$

$$A(\vec{x}^* + \alpha\vec{c}) = \vec{b} \left(\text{for any } \alpha \in \mathbb{R}\right)$$

We say vectors $\{\vec{a}_1 \dots \vec{a}_n\}$ are <u>linearly independent</u> is not lin. dependent.

The only scalars $c_1 \dots c_n$ such that

$$\sum_{i=1}^{n} c_i \vec{a}_i = \vec{0}$$

are $c_1 = c_2 = \dots = c_n = 0$

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_n \vec{a}_n = \vec{0}$$

$$\underbrace{[\vec{a}_1 \ \dots \ \vec{a}_n]}_{A} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{\vec{c}} = \vec{0}$$

$A\vec{c} = \vec{0}$ has a unique solution of $\vec{c} = \vec{0}$

How do we test if $\vec{a}_1 \ \vec{a}_2 \dots \vec{a}_n$ are lin. dep. or lin. indep.?

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix} \vec{x} = \vec{0}$$

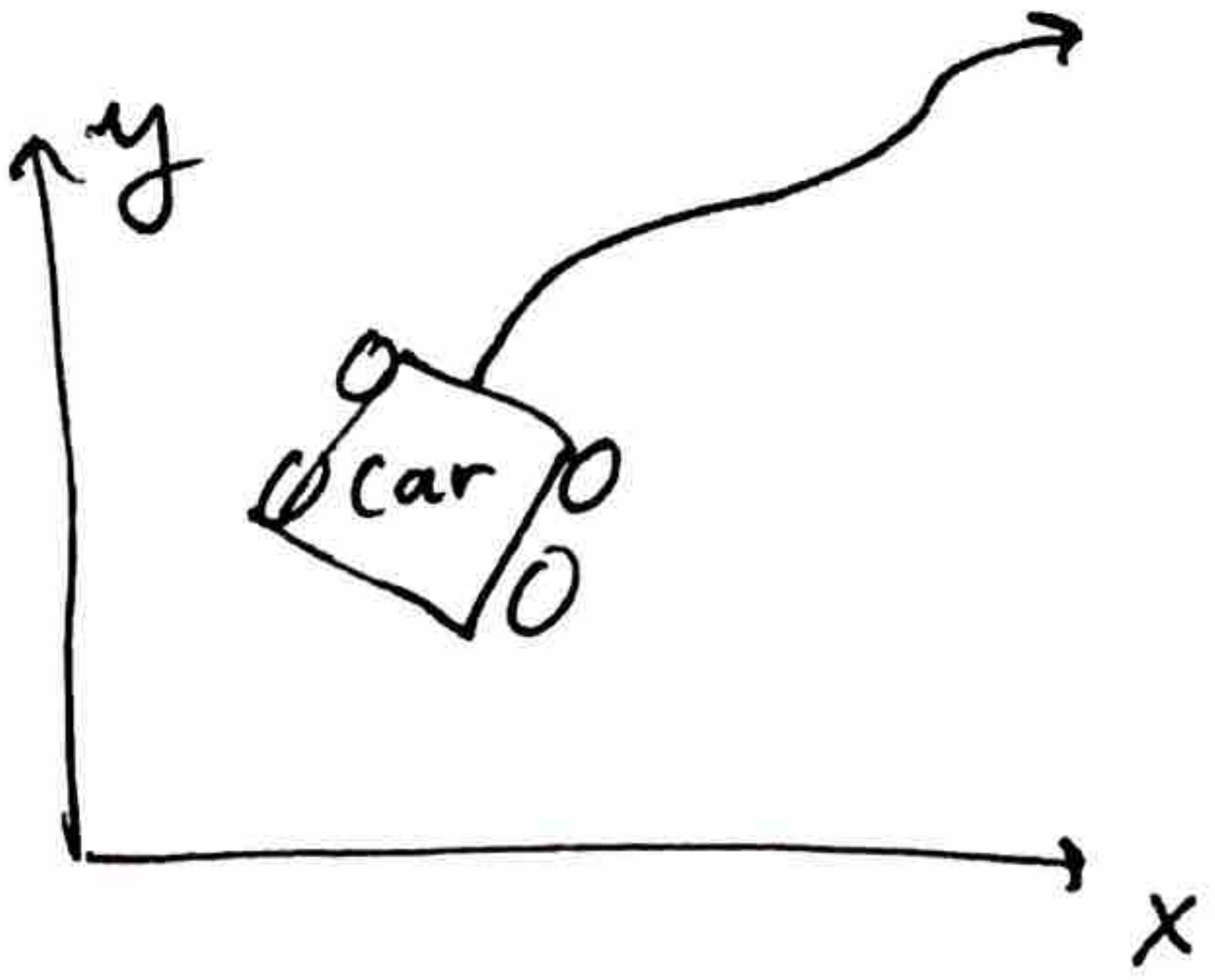
$$A\vec{x} = \vec{0}$$

→ if unique solution ⇒ linearly independent

→ if inf. solutions ⇒ lin. dependent

→ if no ~~solution~~ ⇒ Never happens because $\vec{x} = \vec{0}$ is alway a sol. to $A\vec{x} = \vec{0}$

State Transformations



x position

y position

V_x velocity x

V_y velocity y

$$\text{state of system} \rightarrow \begin{bmatrix} x(t) \\ y(t) \\ V_x(t) \\ V_y(t) \end{bmatrix} = \vec{s}(t)$$

$$= f(x(t), y(t), V_x(t), V_y(t))$$

$$x(t+1)$$

$$y(t+1) = g(x(t), y(t), V_x(t), V_y(t))$$

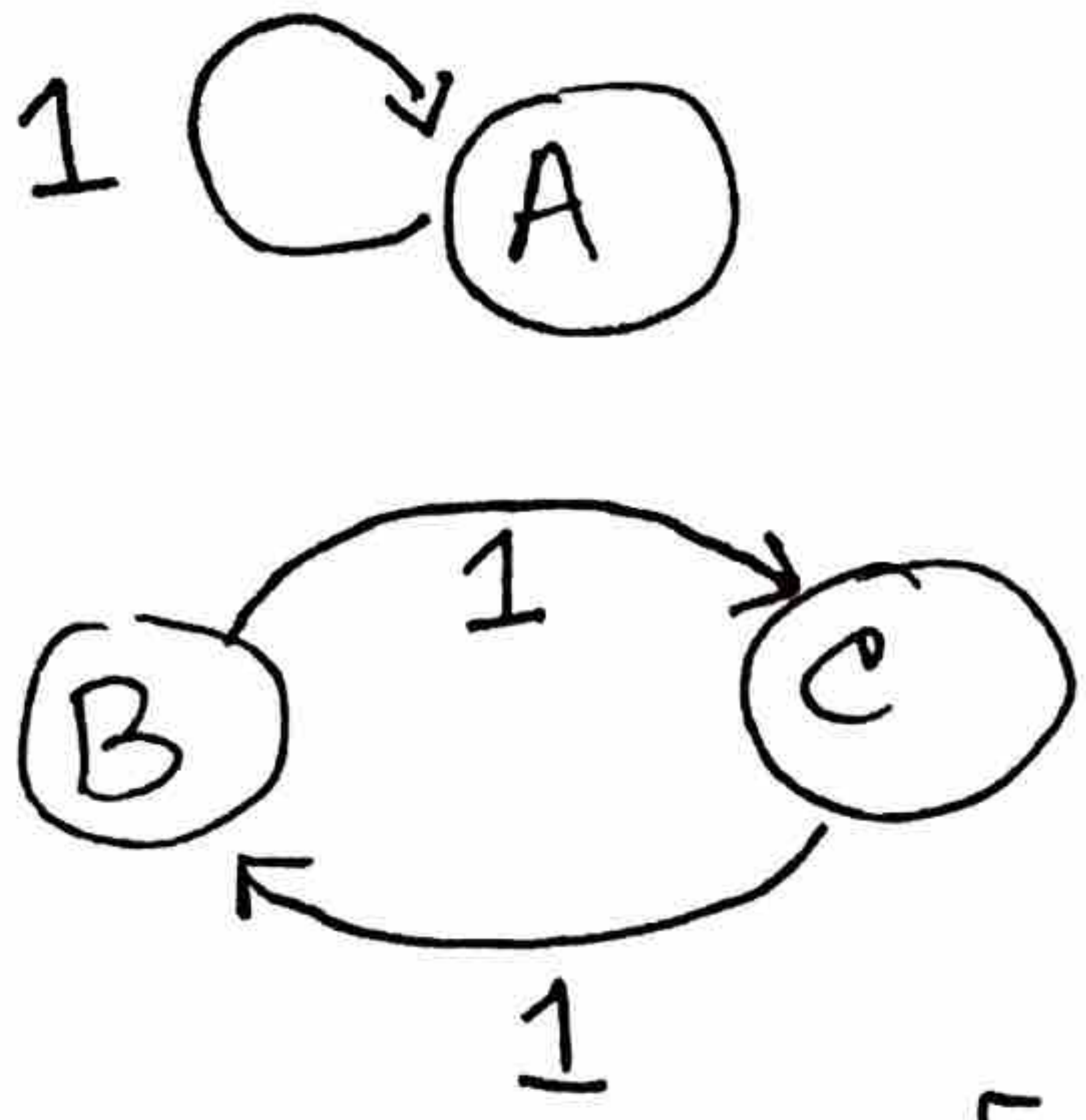
f, g are linear (assumption!)

$$\hookrightarrow x(t+1) = a_1 x(t) + a_2 y(t) + a_3 V_x(t) + a_4 V_y(t)$$

$$\begin{bmatrix} x(t+1) \\ y(t+1) \\ V_x(t+1) \\ V_y(t+1) \end{bmatrix} = \vec{s}(t+1) = \underbrace{A}_{\text{state transition matrix}} \vec{s}(t)$$

Represents how system evolves over time

System of pumps and reservoirs



$$\underbrace{\vec{x}(t)}_{\text{state vector}} = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

water in tank A

Run pumps:

$x_A(t+1) = x_A(t)$
$x_B(t+1) = x_C(t)$
$x_C(t+1) = x_B(t)$

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\substack{\text{State transition} \\ \text{matrix} \\ Q}} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

$$\vec{x}(t+1) = Q\vec{x}(t)$$

But what about 2 time steps from now?

$$\vec{x}(t+2) = Q\vec{x}(t+1) = \underbrace{Q(Q}_{P}\vec{x}(t))$$

$$\vec{x}(t+2) = P\vec{x}(t)$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{identity matrix!} \\ I}}$$

$$I\vec{x} = \vec{x}$$

Matrix - Matrix Multiplication

"stacked" matrix - vector multiplication

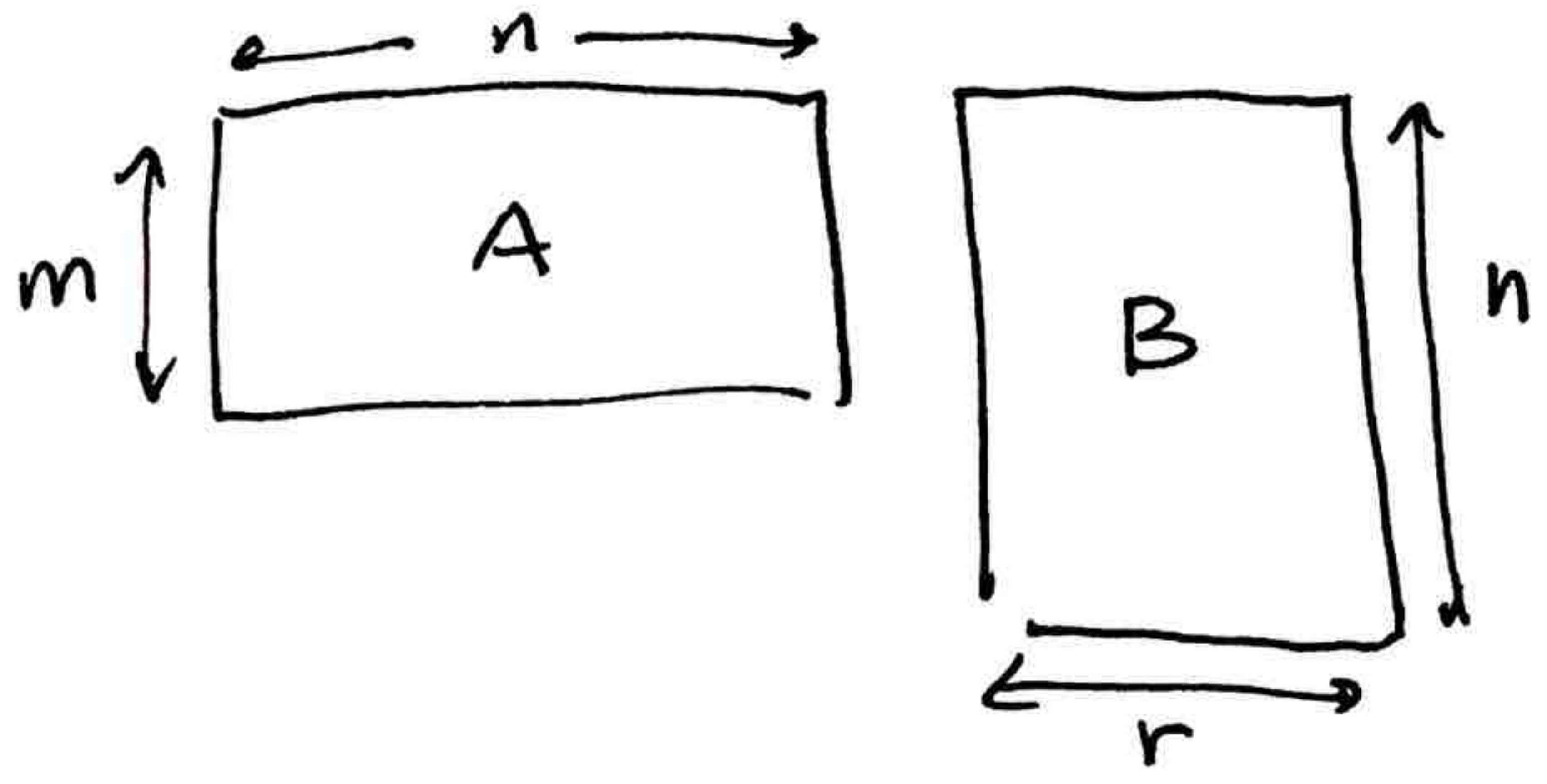
$$A \cdot B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Bigg[\underbrace{\begin{matrix} b_{11} \\ b_{21} \end{matrix}}_{\vec{b_1}} \quad \underbrace{\begin{matrix} b_{12} \\ b_{22} \end{matrix}}_{\vec{b_2}} \Bigg] = A \left[\vec{b_1} \quad \vec{b_2} \right]$$

$$= \left[A\vec{b_1} \quad A\vec{b_2} \right]$$

$$AB = \Bigg[\underbrace{\begin{matrix} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \end{matrix}}_{A\vec{b_1}} \qquad \underbrace{\begin{matrix} a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{12} + a_{22}b_{22} \end{matrix}}_{A\vec{b_2}} \Bigg]$$

Generically:

$A \in \mathbb{R}^{m \times n}$ $\quad B \in \mathbb{R}^{n \times r}$



$$= A \left[\vec{b_1} \quad \vec{b_2} \quad \cdots \quad \vec{b_r} \right]$$

$$= \underbrace{\left[A\vec{b_1} \quad A\vec{b_2} \cdots \quad A\vec{b_r} \right]}_{? = r \text{ column}} \updownarrow m$$

\# columns of A
matches # of rows of B

$$\underset{m \times n}{A} \quad \underset{n \times r}{B} = \underset{m \times r}{C}$$

$$\begin{bmatrix} 1 & 0 \\ 10 & 100 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 310 & 420 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 10 & 100 \end{bmatrix} = \begin{bmatrix} 21 & 200 \\ 43 & 400 \end{bmatrix}$$

NOT EQUAL!

$AB \neq BA$ Matrix-matrix multiplication does not commute!

common mistake

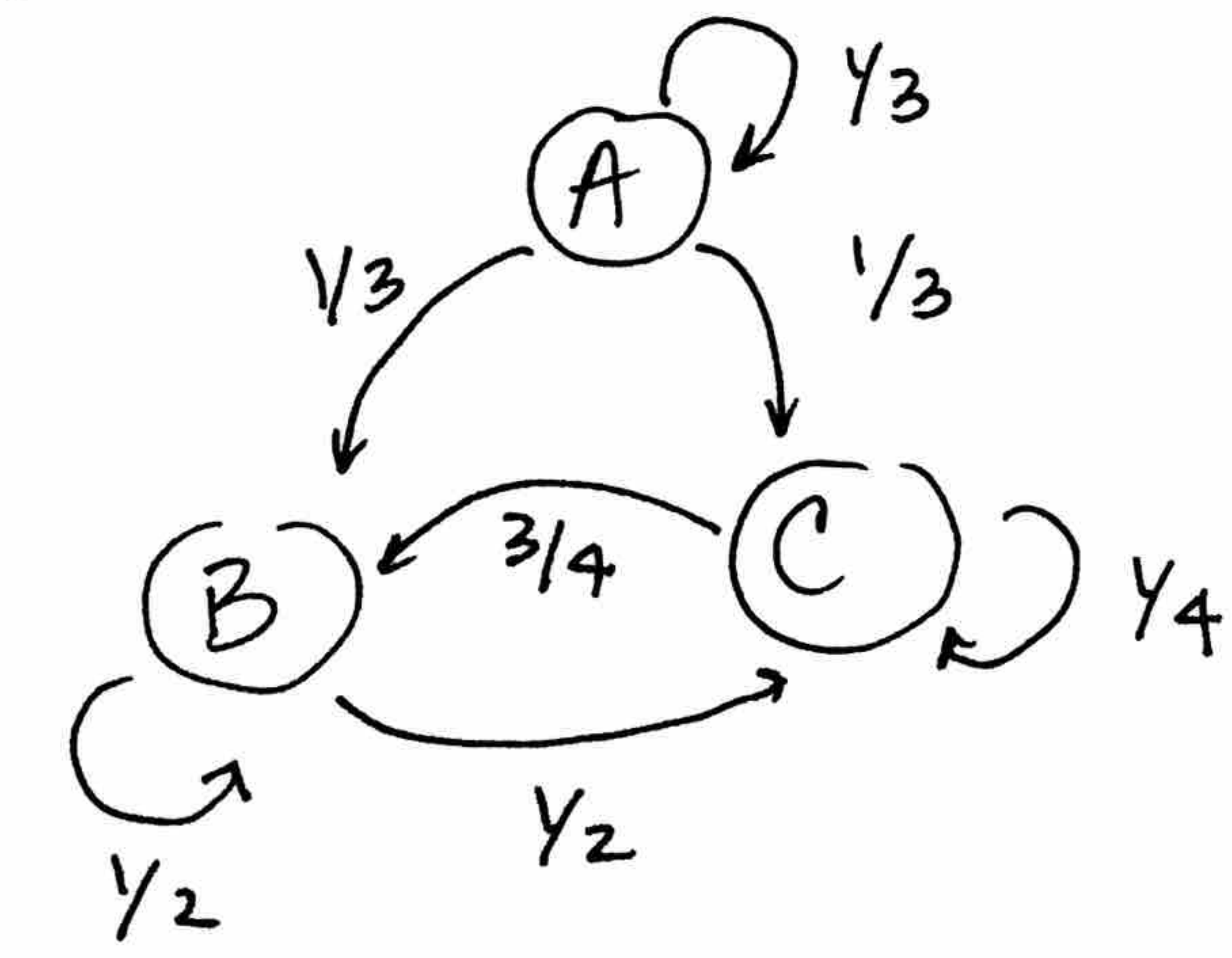
$A = B$ I want to mult. by C

"left multiply": $CA = CB$

"right multiply": $AC = BC$

$\cancel{AC = CB}$ Not allowed b/c $AB \neq BA$

Another Example



$$Q = \begin{bmatrix} 1/3 & 0 & 0 \\ 1/3 & 1/2 & 3/4 \\ 1/3 & 1/2 & 1/4 \end{bmatrix}$$