EECS 16A
Lecture 1B
June 30, 2020
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Today:
- Matrix inversion

$$\left.\begin{aligned} f(x) &= 2x \\ g(x) &= \tfrac{1}{2}x \end{aligned}\right\} \text{inverses}$$

$$g(f(x)) = \tfrac{1}{2}(2x) = x$$

$$f(g(x)) = 2(\tfrac{1}{2}x) = x$$

Today: extend this idea to matrices!

Yesterday: AB

$AB \neq BA$ does not commute generally

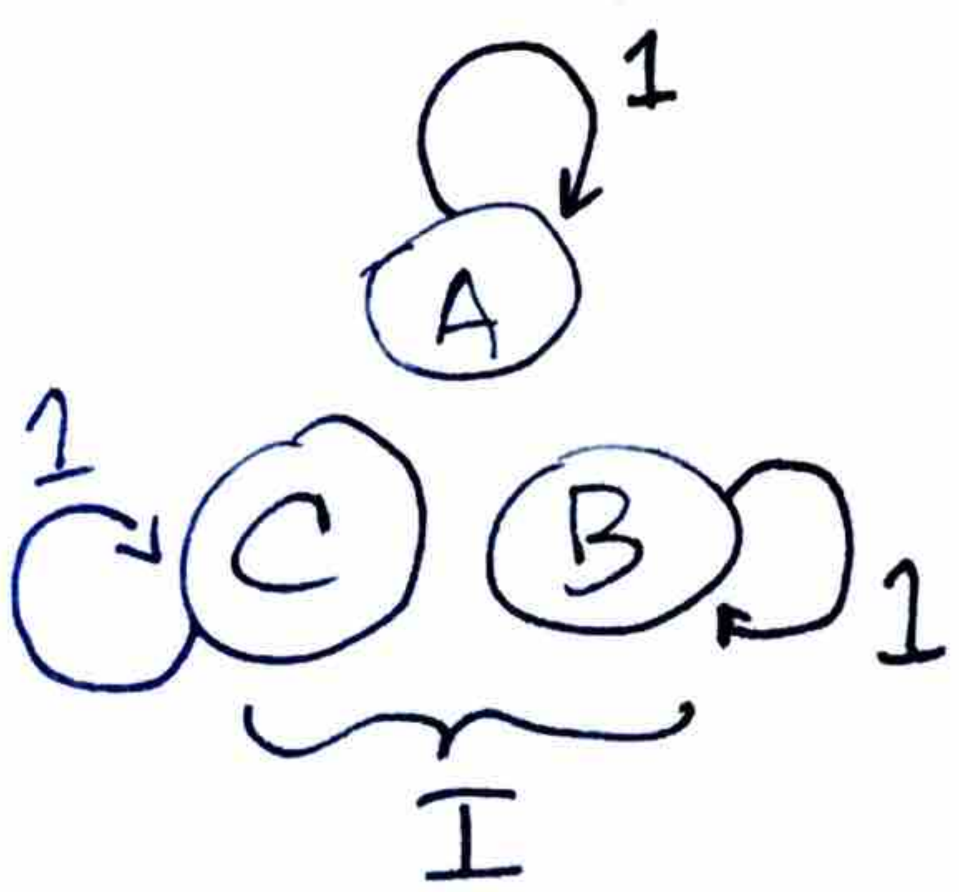
$A(BC) = \underbrace{(AB)}_{D}C = DC$ Associative property holds!

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$I \leftarrow$ identity matrix
square matrix with ones on diagonal

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A$$



$$IA = A$$

$$AI = A$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

1. Run Q once $\quad \vec{x}(1) = Q\vec{x}(0)$
2. Run R once $\quad \vec{x}(2) = R\vec{x}(1)$

$$\vec{x}(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = ? = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

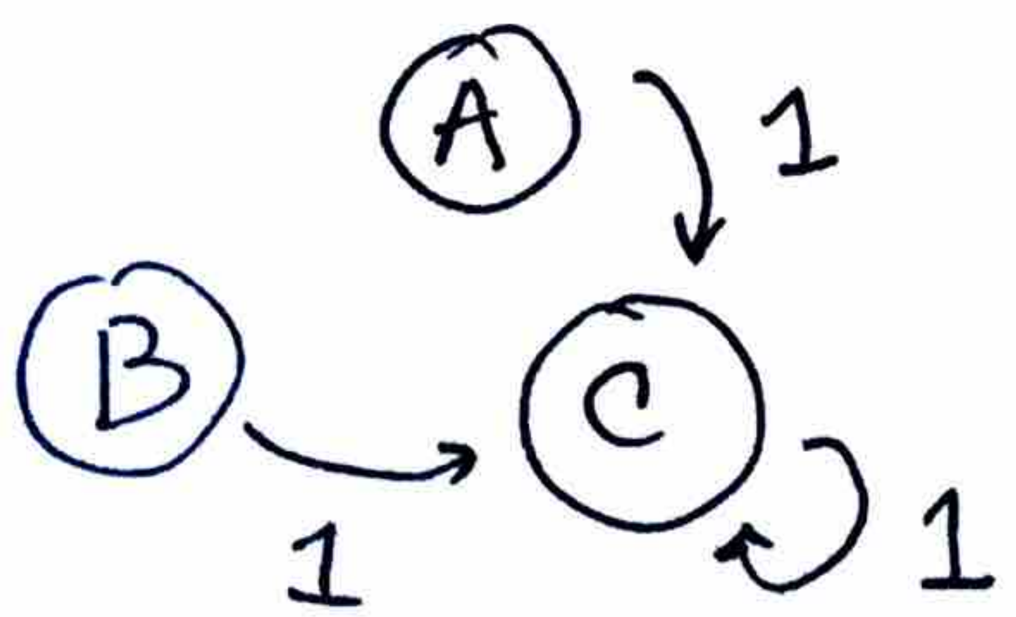
$$\vec{x}(2) = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \\ 2 \end{bmatrix}$$

What pump system would give us $\vec{x}(2)$ directly from $\vec{x}(0)$?

$$\begin{aligned} \vec{x}(2) &= R\vec{x}(1) \\ &= R(Q\vec{x}(0)) \\ &= \underbrace{(RQ)}_{\text{new pumps}}\vec{x}(0) \end{aligned}$$

Suppose we have pumps $\vec{x}(t+1) = Q\vec{x}(t)$.
How do we go backwards and return to $\vec{x}(t)$ from $\vec{x}(t+1)$?

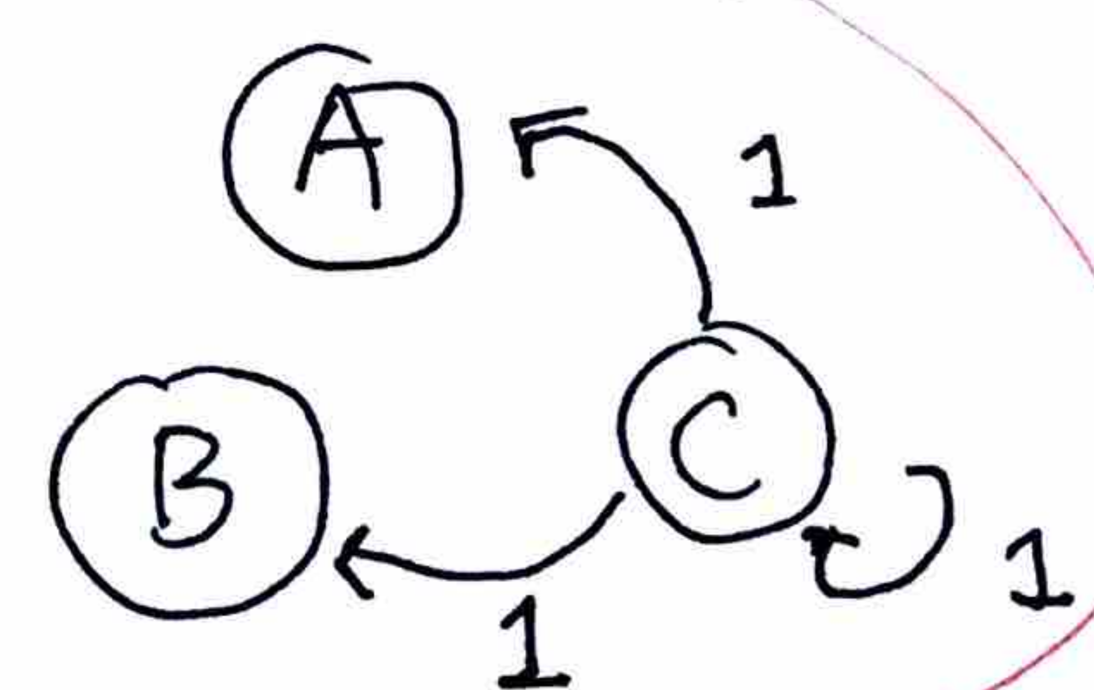
Does not take us backwards



start A:1 C:1
B:1

reverse arrows ??

A: 0
B = 0
C = 3



A: 3
B: 3
C: 3

Pumps that create water or remove ~~ma~~ water are called <u>non-conservative</u> systems.

If the total water stays the same, it's a <u>conservative</u> system.

Goal: Want a matrix R such that

$$\vec{x}(t) = R\vec{x}(t+1)$$

We said $\vec{x}(t+1) = Q\vec{x}(t)$

$$\begin{aligned}\vec{x}(t) &= R\vec{x}(t+1) \\ &= R(Q\vec{x}(t)) \\ \vec{x}(t) &= \underbrace{(RQ)}_{I}\vec{x}(t)\end{aligned}$$

$$RQ = I$$

$$\begin{aligned}\vec{x}(t+1) &= Q\vec{x}(t) \\ &= Q(R\vec{x}(t+1)) \\ \vec{x}(t+1) &= \underbrace{(QR)}_{I}\vec{x}(t+1)\end{aligned}$$

$$QR = I$$

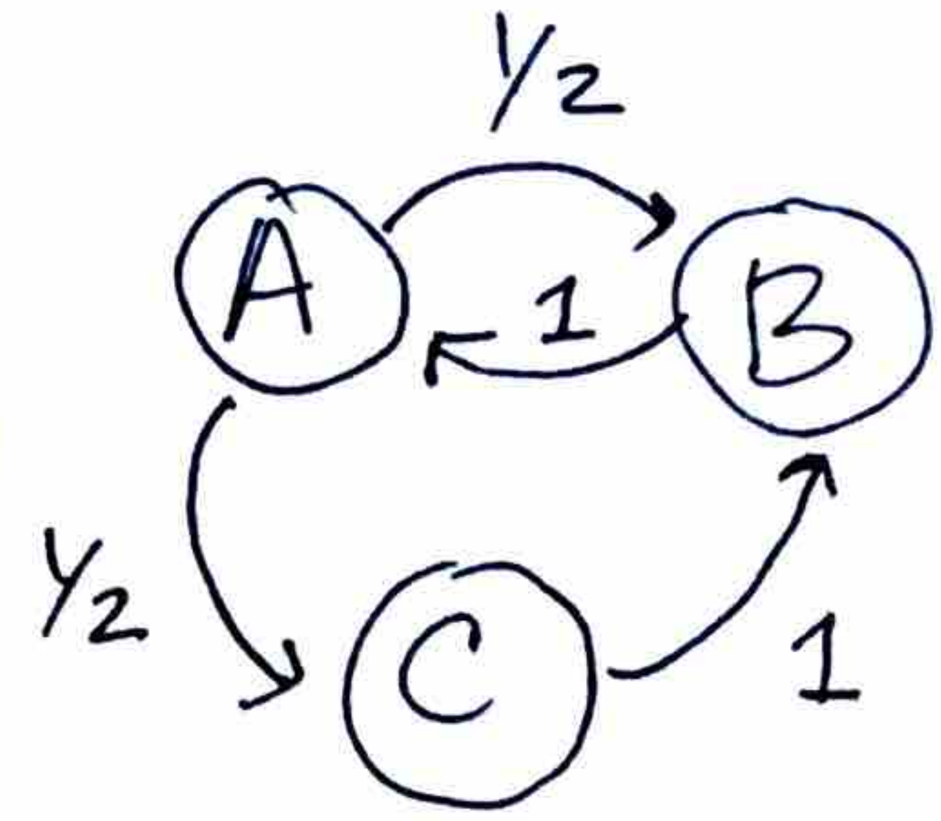
$\boxed{\text{Definition:}}$ If P and Q are both square matrices in $\mathbb{R}^{n \times n}$, then matrix P is said to be the inverse of Q if

$$P \cdot Q = I \quad \text{and} \quad Q \cdot P = I$$

$$PQ = QP = I$$

We'll denote the inverse of Q (in this case is P) as Q^{-1} (when Q^{-1} exists)

EX) $Q = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix}$



$$\vec{x}(t+1) = Q\,\vec{x}(t)$$

want P such that

$$\underset{\text{known}}{Q} \cdot \underset{\text{unknown}}{P} = \underset{\text{known}}{I}$$

Does this remind of anything?

$$A\vec{x} = \vec{b}$$

knowns unknown

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix}}_{Q} \begin{bmatrix} \underbrace{\begin{matrix} p_{11} \\ p_{21} \\ p_{31} \end{matrix}}_{\vec{p}_1} & \underbrace{\begin{matrix} p_{12} \\ p_{22} \\ p_{32} \end{matrix}}_{\vec{p}_2} & \underbrace{\begin{matrix} p_{13} \\ p_{23} \\ p_{33} \end{matrix}}_{\vec{p}_3} \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}}_{Q\vec{p}_1} & \underbrace{\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}}_{Q\vec{p}_2} & \underbrace{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}}_{Q\vec{p}_3} \end{bmatrix}$$

$$Q\vec{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad Q\vec{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad Q\vec{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Observation: That Gaussian elimination process does not depend on $\vec{b}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1/2 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 1 \\ 1/2 & 0 & 0 & 0 \end{array}\right]$$

Idea! Instead of doing G.E. 3 times, lets do all 3 at once!

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

swap $R_1 R_2$

$$\left[\begin{array}{ccc|ccc} 1/2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R1 \leftarrow 2R1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R3 \leftarrow R3 - 1/2 R1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right]$$

$R3 \leftarrow -R3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$R1 \leftarrow R1 - 2R3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\underbrace{\vec{p}_1 \quad \vec{p}_2 \quad \vec{p}_3}_{P}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$p_{11} = 0$

$p_{21} = 1$

$p_{31} = 0$

$$Q^{-1} = P = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$2x = 4$

$\frac{1}{2}(2x) = \frac{1}{2}(4)$

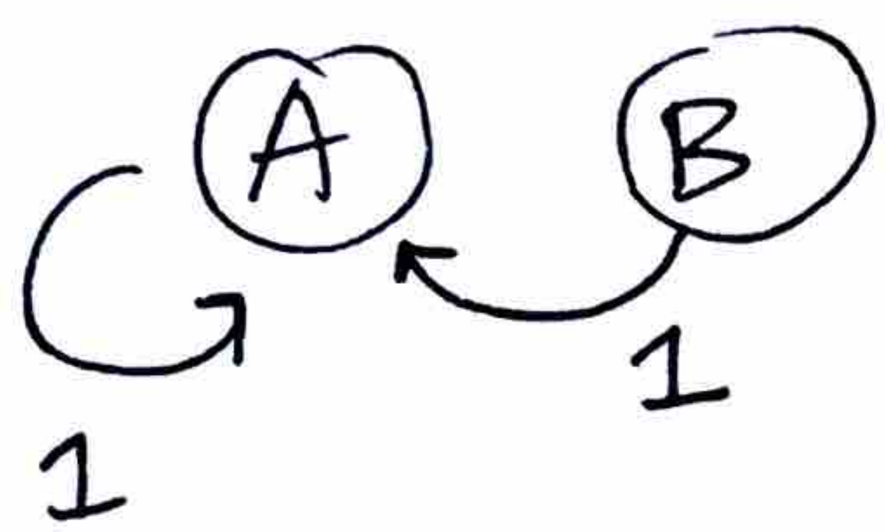
$x = 2$

A/B Not defined

$f(x) = 2x \rightarrow g(x) = \frac{1}{2}x$

$f(x) = 0x$

ex)



$Q = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$[\, Q \mid I \,]$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No solution $\Rightarrow$ no inverse

Q is not invertible

Generally if we want to invert Q

$$\underset{\text{create augmented matrix}}{[\, Q \mid I \,]} \xrightarrow[\text{G.E.}]{} \underbrace{[\, I \mid Q^{-1} \,]}_{\text{if there is an inverse}}$$

$QP = I$ does this mean $PQ = I$?

known, unknown, known

$\boxed{\text{THM}}$ If $QP = I$ and $RQ = I$ then $R = P$.

right inverse, left inverse

"The right and left inverses of Q are the same"

Proof

We know $QP = I$
$RQ = I$

want to show: $R = P$

Idea 1: $\cancel{Q}P = R\cancel{Q}$

cancel Q (very tempting)
No notion of division
Not allowed!

Try again $QP = I$

left multiply by R

$R(QP) = RI$

$\underbrace{(RQ)}_{I} P = R$

$IP = R$

$P = R$ ✓

Multiply on right

$(QP)R = IR$

$\underbrace{QP}_{I}R = R$

$IR = R$

$R = R$

Inversion $\longleftrightarrow$ unique solution $\longleftrightarrow$ linear independence dependence of columns

$\updownarrow$

gaussian elimination

$\boxed{\text{THM}}$ If the columns of A are linearly dependent, then the matrix A is not invertible.

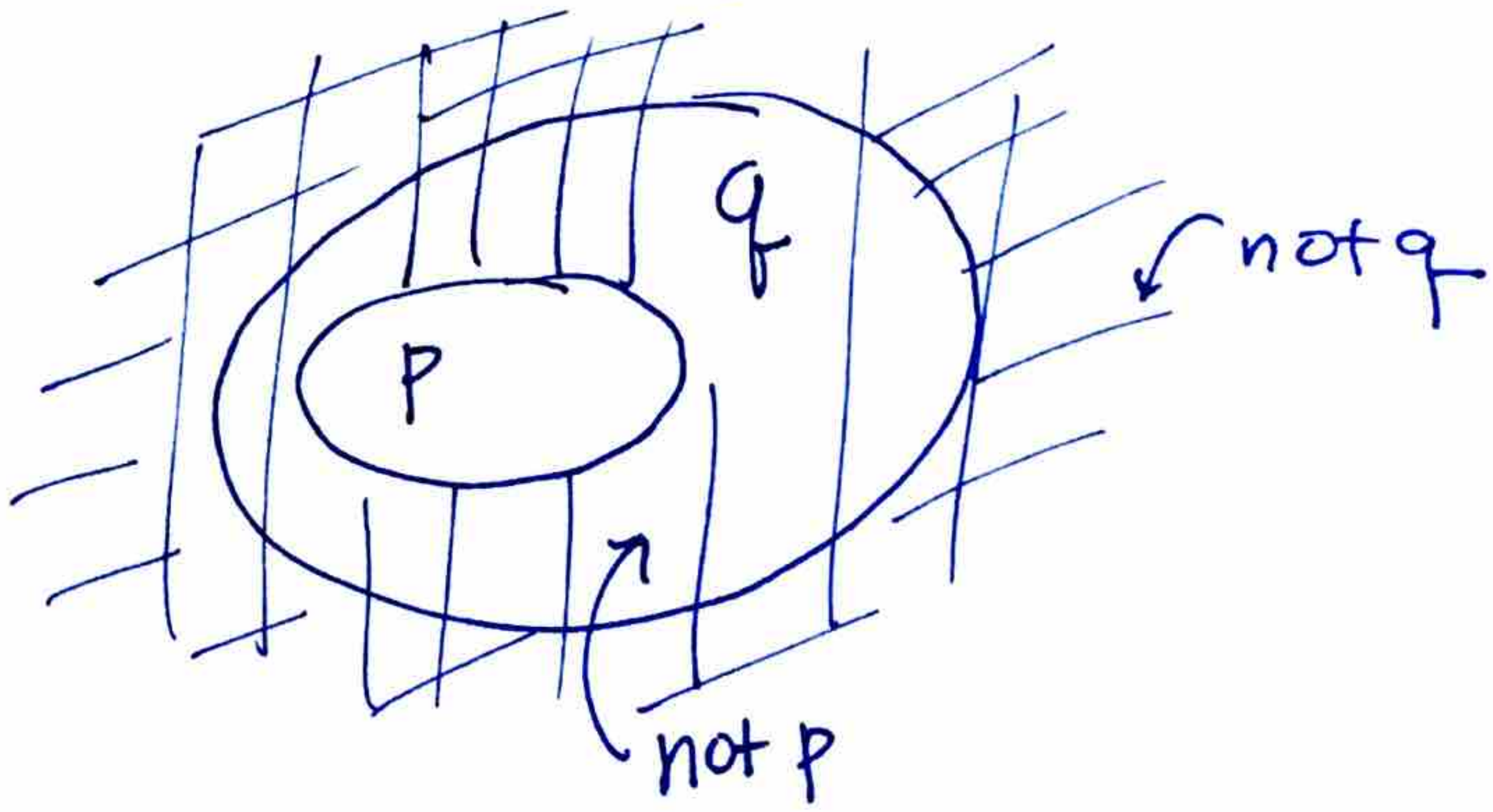
Aside: $p \Rightarrow q$

if it's raining then there are clouds

if there are no clouds then its not raining

"contra positive"

not $q \Rightarrow$ not p



Equivalent THM:

if the matrix A is invertible then the columns of A are linearly independent.

Proof

Start: columns A are linearly dependent

End: matrix A is <u>not</u> invertible

Assume A is invertible so A^{-1} exists

let's call columns $\vec{a}_1\ \vec{a}_2 \dots \vec{a}_n$ $\quad A = [\vec{a}_1 \dots \vec{a}_n]$

there exist c_i's not all zero such that

$$c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n = \vec{0}$$

$$[\vec{a}_1 \dots \vec{a}_n]\underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{\vec{c}} = \vec{0}$$

$$A\vec{c} = \vec{0}$$

we know $\vec{c} \neq \vec{0}$

left mult. by A^{-1}

$$(A^{-1}A)\vec{c} = A^{-1}\vec{0}$$

$$I\vec{c} = A^{-1}\vec{0}$$

$$I\vec{c} = \vec{0}$$

$$\vec{c} = \vec{0}$$

contradiction