EECS 16A
July 1, 2020
Lecture 1C

Topics
Inversion,
Vector space/subspace
Basis

Announcements:
1) HW1A is due tonight
2) HW1B will be up today (due on Monday)
3) No HW party / OH on Friday (rescheduling)

Inversion: Undoing linear transformation

$f(x) = 5x$ ← reversible
$g(x) = \sin x$ ← irreversible

x		$f(x)$
1	$\rightleftarrows$	5
2	$\rightleftarrows$	10

x		$g(x)$
30°	→	$\frac{1}{2}$
150°	↗	

If $Ax_1 = Ax_2 = b$
$x_1 \neq x_2$
A should not be invertible

Theorem: If A is Invertible, then there is unique solution to $A\bar{x}=\bar{b}$ for any $\bar{b}$.

Proof: A^{-1} exists $\Rightarrow$ $AA^{-1} = I$ (I)

$A^{-1}A = I$ (II)

a) Can't use $A\bar{x}=\bar{b}$, unless we know there is at least one sol^n

$$\boxed{\begin{aligned} &\times\ A\bar{x} = \bar{b} \\ &A^{-1}\bar{A}\bar{x} = A^{-1}\bar{b} \\ &\Rightarrow \bar{x} = A^{-1}\bar{b} \end{aligned}}$$

(I) $\times \bar{b} \Rightarrow$

$$AA^{-1} \times \bar{b} = I \times \bar{b}$$

$$\Rightarrow A(A^{-1}\bar{b}) = \bar{b}$$

$\bar{x}_0 = A^{-1}b$ is a sol^n

b) Let's assume $\bar{x}_1$ is another sol^n.

$x_0 \neq x_1$ $\quad A\bar{x}_1 = \bar{b}$

$$\Rightarrow A^{-1}A\bar{x}_1 = A^{-1}\bar{b}$$

$$\Rightarrow Ix_1 = A^{-1}b \Rightarrow \bar{x}_1 = A^{-1}\bar{b} = x_0$$

Contradiction

Target: $A\bar{x}=\bar{b}$ has a unique sol^n.

a) sol^n exists

b) sol^n is unique. $= A^{-1}\bar{b}$

If $A \in \mathbb{R}^{n \times n}$ is invertible

$\Leftrightarrow$ Linearly indep columns

$\Leftrightarrow$ " " rows

$\Leftrightarrow$ $A\bar{x} = \bar{b}$ has a unique sol^n : $A^{-1}\bar{b}$

$\Leftrightarrow$ $A\bar{x} = \bar{0}$ has a " " : $\bar{0}$

$\rightarrow$ trivial solution

Solution to $A\bar{x} = \bar{0}$ $\leftarrow$ Nullspace (A)

Vector space   Note 7

A vector space V is a set of vectors satisfying the following properties:

Vector addition: $\bar{u}, \bar{v}, \bar{w} \in V$

I) $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$

II) $\bar{u} + \bar{v} = \bar{v} + \bar{u}$

III) There exists $\bar{0} \in V$, so that $\bar{0} + \bar{v} = \bar{v}$

IV) $\bar{v} + (-\bar{v}) = 0$

v) Closure under vector addition :

$\bar{u} \in V$, $\bar{v} \in V$, then

$\bar{u}+\bar{v} \in V$

Scalar multiplication: α, β are scalars

i) $\alpha(\beta\bar{v}) = (\alpha\beta)\bar{v}$

ii) $1.\bar{v} = \bar{v}$

iii) $\alpha(\bar{u}+\bar{v}) = \alpha\bar{u}+\alpha\bar{v}$

iv) $(\alpha+\beta)\bar{v} = \alpha\bar{v}+\beta\bar{v}$

v) Closure under scalar multiplication

If α is a scalar, $\bar{v} \in V$ then

$\alpha\bar{v} \in V$

Examples of vector space.:

$\mathbb{R}^2, \mathbb{R}^3 \ldots\ldots$ $\mathbb{R}^n \rightarrow$ canonical space

Dimension of $\mathbb{R}^n = n$

Basis: For a vector space V, a set of vectors $\{\bar{v}_1, \dots \bar{v}_n\}$ is called a basis for V, if it satisfies:

i) $\bar{v}_1, \dots, \bar{v}_n$ are lin indep

ii) $V = \text{span}\{\bar{v}_1, \dots v_n\}$

★ What is the dimension of V if $\{\bar{v}_1, \bar{v}_2, \dots \bar{v}_n\}$ is a basis for V?

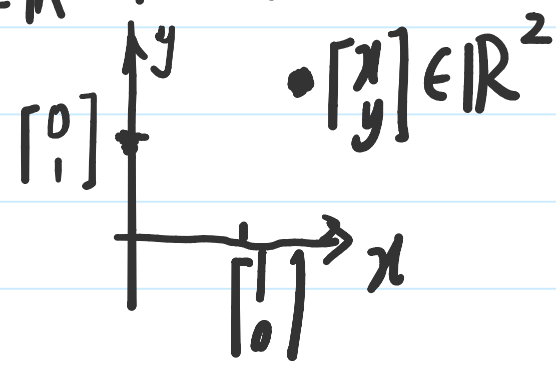
$\underline{n}$

<u>Ex 1</u> Is $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ a basis for $\mathbb{R}^2$

i) linearly indep ✓

ii) $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \mathbb{R}^2$? ✓

$$\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}$$



Yes

Ex 2 Is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^2$?

i) linearly indep ✓

ii) $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\rightarrow$ can you solve for c_1 & c_2 uniquely?

$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & 1 & y \end{array}\right]$

Yes

$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$

Takeaways:

* choice of basis is not unique
* We need n linearly indepedent vectors to form a basis for an n-dimensional space.
* $\bar{v} \in V$ can be written as a unique linear combination of basis vectors $\{\bar{v}_1, \bar{v}_2, \dots \bar{v}_n\}$; i.e.

$$\bar{v} = \sum \alpha_i \bar{v}_i$$

Ex 3

Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

i) Linearly indep ✓

ii) We need 3 vectors to span $\mathbb{R}^3$ ✗

No

Is $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ a basis for any other vector space?

$\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ can span a plane that $\in \mathbb{R}^3$

[★ $\mathbb{R}^2$ has only two element vectors, so these two vectors in $\mathbb{R}^3$ cannot span $\mathbb{R}^2$)

$$\underline{V} = \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

$$= \left\{c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} \mid c_1 \,\&\, c_2 \text{ are scalars}\right\}$$

$$= \left\{\begin{bmatrix}c_1\\c_1+c_2\\c_2\end{bmatrix} \mid c_1, c_2 \text{ are scalars}\right.$$

V is a two dimensional space that has a basis of $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

Distinguishing span & basis:

$$\text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\} \Rightarrow \mathbb{R}^2$$

$$\text{Basis for } \mathbb{R}^2: \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} \leftarrow \text{lin indep}$$

$$\text{or } \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$$

$$\text{or } \left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}\right\}$$

etc.

Subspace: Note 8

U consists of a subspace of a vector space V if the following are satisfied

i) closed under vector addition

If, $\bar{u}_1, \bar{u}_2 \in U$, then

$\bar{u}_1 + \bar{u}_2 \in U$

ii) Closed under scalar multiplication:

If $u \in U$, and α is a scalar then $\alpha \bar{u} \in U$

iii) $\bar{0} \in U$

* Any subspace is a vector space

Example :

Show that, $U = \left\{ \begin{bmatrix} m \\ n \\ m+n \end{bmatrix} \;\middle|\; m,n \to \text{scalars} \right\}$

is a subspace of $\mathbb{R}^3$

$$\bar{u} = \begin{bmatrix} m \\ n \\ m+n \end{bmatrix} = m\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + n\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in U$$

<u>Property 1:</u>

$$\bar{u}_1 = \begin{bmatrix} m_1 \\ n_1 \\ m_1+n_1 \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} m_2 \\ n_2 \\ m_2+n_2 \end{bmatrix}$$

$$\bar{u}_1 + \bar{u}_2 = \begin{bmatrix} m_1+m_2 \\ n_1+n_2 \\ m_1+m_2+n_1+n_2 \end{bmatrix}$$

$$= (m_1+m_2)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (n_1+n_2)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in U$$

Property 2 For any scalar α

$$\alpha\bar{u} = \begin{bmatrix} \alpha m \\ \alpha n \\ \alpha m + \alpha n \end{bmatrix}$$

$$= \alpha m \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha n \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in U$$

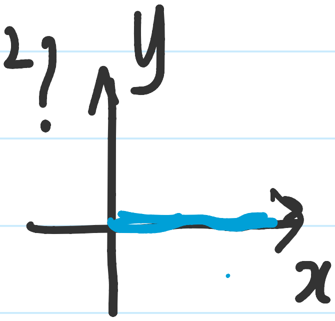
Property 3 If m, n are zero

$$\bar{u} = \begin{bmatrix} 0 \\ 0 \\ 0+0 \end{bmatrix} = \bar{0} \in U$$

U is a subspace of $\mathbb{R}^3$ ✓✓

Example:

Are these subspaces of $\mathbb{R}^2$?



★ $V = \operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}\right\}$

is a subspace of $\mathbb{R}^2$

★ $V = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}\right\}$ Not a subspace

i) $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}4\\0\end{bmatrix} \notin V$

ii) $3\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}9\\0\end{bmatrix} \notin V$

iii) $\begin{bmatrix}0\\0\end{bmatrix} \notin V$