EECS 16A
July 2, 2020
Lecture 1D

Topics:
- Nullspace
- Columnspace

Announcements:
1) Friday OH happening today
2) Friday HW party moved to Monday

* For an n-dimensional space, we need n basis vectors.

$A \in \mathbb{R}^{m \times n}$ For $A\bar{x} = \bar{b}$

$\bar{x} \in \mathbb{R}^n$ Transformation

$\bar{b} \in \mathbb{R}^m$ $\mathbb{R}^n \rightarrow \mathbb{R}^m$

If, $\bar{x} = \begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{R}^2$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 2}$

$\bar{b} = A\bar{x} = \begin{bmatrix} m \\ n \\ m+n \end{bmatrix} \in \mathbb{R}^3 \leftarrow$ Dim = 2

Lecture 1C

Nullspace: All solutions for $A\bar{x}=0$ is called the nullspace of A.

For $A \in \mathbb{R}^{m\times n}$, $N(A) = \{x \in \mathbb{R}^n \mid A\bar{x}=0\}$

Ex 1: Show that $N(A)$ is a subspace of $\mathbb{R}^n$, where $A \in \mathbb{R}^{m\times n}$

i) If $\bar{x}_1 \in N(A)$, $\bar{x}_2 \in N(A)$

$$A\bar{x}_1 = \bar{0} \quad (\text{I}) \qquad A\bar{x}_2 = \bar{0} \quad (\text{II})$$

(I) + (II)

$$A\bar{x}_1 + A\bar{x}_2 = \bar{0}$$

$$\Rightarrow A(\bar{x}_1+\bar{x}_2) = \bar{0}$$

So $(\bar{x}_1+\bar{x}_2) \in N(A)$

ii) If $\bar{x} \in N(A)$

$$\Rightarrow A\bar{x} = \bar{0}$$

$$\Rightarrow \alpha A\bar{x} = \bar{0} \quad \Rightarrow A(\alpha\bar{x}) = 0$$

$\Rightarrow \alpha\bar{x} \in N(A)$, α is a scalar

iii) $A(\bar{0}) = \bar{0}$, $\bar{0} \in N(A)$

$N(A)$ is a subspace of $\mathbb{R}^n$

* Can be also proven for columnspace

Columnspace/range: Span of columns of A.

If $A = \begin{bmatrix} | & & | \\ \bar{a}_1 & & \bar{a}_n \\ | & & | \end{bmatrix} \in \mathbb{R}^{m\times n}$, columnspace

$= C(A) = \text{Range}(A)$

$= \{A\bar{x} \mid \bar{x} \in \mathbb{R}^n\}$

$$A\bar{x} = \begin{bmatrix} | & & | \\ \bar{a}_1 & \cdots & \bar{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$= [\bar{a}_1 x_1 + \bar{a}_2 x_2 + \cdots]$

$=$ linear comb of $\{\bar{a}_1 \cdots \bar{a}_n\}$

For all values of $\bar{x}$

$A\bar{x} \rightarrow \text{span}\{\bar{a}_1, \ldots \bar{a}_n\}$

$A\bar{x} \rightarrow C(A) \in \mathbb{R}^m$

Lecture 0D

Takeaway:

$$\left.\begin{array}{l} \text{If } \bar{x} \in N(A),\ A\bar{x} = \bar{0} \\ \text{If } \bar{x} \notin N(A),\ A\bar{x} \neq \bar{0} \end{array}\right\} A\bar{x} \rightarrow C(A)$$

Ex 2 $\quad A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 4 & 6 \end{bmatrix}$

Find $N(A)$, $C(A)$ & their basis/dim.

Nullspace: $A\bar{x} = \bar{0}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 2 & 4 & 4 & 6 & 0 \end{array}\right]$$

lin indep columns

$R_2' \leftarrow R_2 - 2R_1$
$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & -2 & -4 & 0 \end{array}\right]$$

$R_2' \leftarrow R_2 / -2$
$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$R_1' \leftarrow R_1 - 3R_2$
$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \end{array}$$

Free variables: x_2, x_4

Basic variables: $x_1 = -2x_2 + x_4$

$x_3 = -2x_4$

$$N(A) = \begin{bmatrix} -2x_2 + x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$= \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right\}$$

Basis for $N(A) = \left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right\}$

$\text{Dim}(N(A)) =$ # basis vectors for $N(A)$

$=$ # free variables

$= 2$

$$C(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}\right\} \quad (\mathbb{R}^2)$$

$$= \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\} \quad (\text{still } \mathbb{R}^2)$$

Basis for $C(A) = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$

$\rightarrow$ columns producing leading entries of 1.

GE : # Free Variables $= \dim(N(A))$
Basic Variables $= \dim(C(A))$

Variables $= n$
$= \#$ columns

Rank-Nullity Theorem:
Rank $= \dim(C(A))$
Nullity $= \dim(N(A))$

If $A \in \mathbb{R}^{m \times n}$ is a linear from $V \to W$, where $V \in \mathbb{R}^n$ & $W \in \mathbb{R}^m$
then
$$\dim(C(A)) + \dim(N(A))$$
$$= \text{Rank}(A) + \text{Nullity}(A)$$
$$= n = \dim(V)$$

Full rank matrix:

GE: All the columns having leading entries of 1 $\rightarrow$ $A\bar{x}=b$ has unique solution

Ex 3 Ipython demo $\rightarrow$ website

LHS		RHS	
$\bar{x}$		$A\bar{x}$	
$N(A)$	$\longrightarrow$	$\{0\}$	$C(A)$
Not of $N(A)$	$\longrightarrow$	$\neq \{0\}$	

Ex1

Ex2

Ex3 $\dim(N(A))=2$ $\qquad$ $\dim(C(A))=1$ $\quad 2+1=3$

3×3 matrix: 3 rows $\rightarrow$ all dependent

3 columns $\rightarrow$ all dependent

Free var $=2$ $\rightarrow \dim(N(A))$

Basic var $=1$ $\rightarrow \dim(C(A))$

Ex 4 $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$

Find Rank & nullity.

Nullspace: $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

\# Free variables = 0 → nullity

\# Basic variables = 2 → rank

Significance of N(A) & C(A):

i) case 1: No solution

No $\bar{x}$ exists that satisfies $A\bar{x} = \bar{b}$

All possible values of $A\bar{x} \rightarrow C(A)$

$\bar{b} \notin C(A)$

ii) Case 2: Unique solution

Free variables = 0

$\dim(N(A)) = 0$

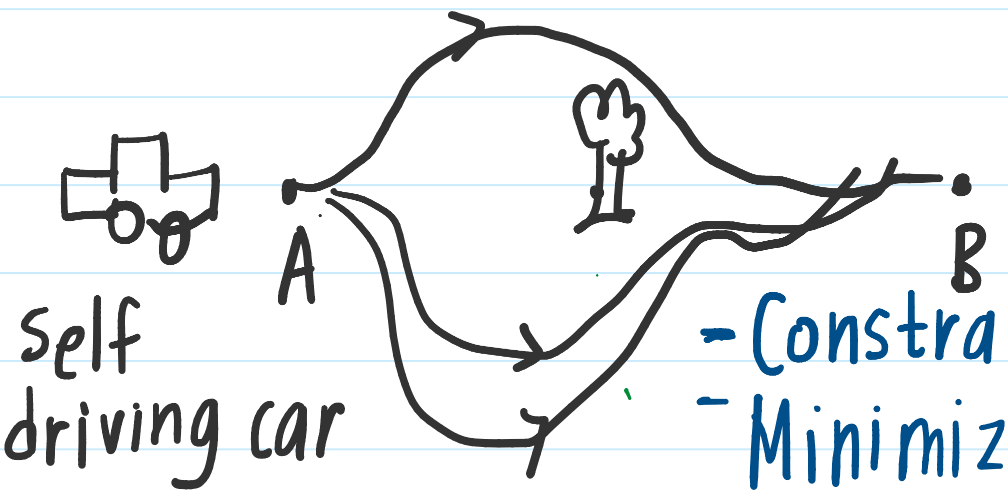
← trivial nullspace

iii) Case 3: Infinite solutions

Free var > 0

$\dim(N(A)) > 0$

* $C(A)$ can tell us if there is a solution to $A\bar{x} = \bar{b}$
* $N(A)$ can characterize all existing solution.



- Constraints
- Minimizing risk
- Minimizing energy

Ex 5 If $\bar{x}_0$ is a solution to $A\bar{x} = \bar{b}$
and $\bar{v} \in N(A)$,
show that $\bar{x}_1 = \bar{x}_0 + \bar{v}$ should be
another solution to $A\bar{x} = \bar{b}$

Given, $\quad A\bar{x}_0 = \bar{b} \quad (I)$

$\qquad\quad\;\; A\bar{v} = \bar{0} \quad (II)$

$(I) + (II) \Rightarrow A(\underbrace{\bar{x}_0 + \bar{v}}) = \bar{b}$

want to show $A\bar{x}_1 = \bar{b}$

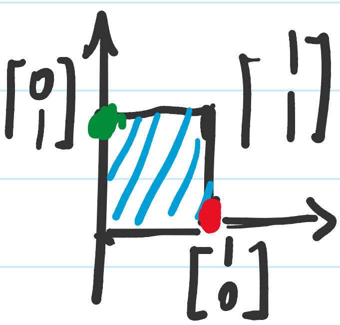
$\bar{x}_1 = \bar{x}_0 + \bar{v}$ is a solution

* If $\bar{x}_0$ is a solution to $A\bar{x} = b$
and $\bar{x}_1 = \bar{x}_0 + \bar{v}$ is another soln
to $A\bar{x} = \bar{b} \Rightarrow \bar{v}$ must belong
in the nullspace of A.

Determinant: For $A \in \mathbb{R}^{n \times n}$, $\det(A)$ is the volume of n-dimensional parallelepiped spanned by columns of A.

→ Det (A) represents the change of volume when transformed by A.

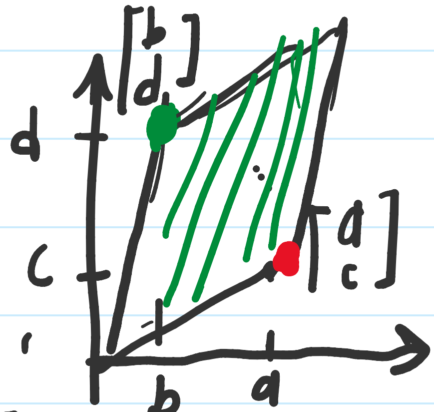
Example: $A \in \mathbb{R}^{2 \times 2}$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$



$\longrightarrow$

$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$,

$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$



Input : $\bar{x}$

Volume = 1

Output: $A\bar{x}$

Volume = ad − bc

Change of volume = $\dfrac{ad-bc}{1}$ = ad-bc

$\det(A) = ad - bc$

For $\mathbb{R}^{2\times2}$, volume $\neq 0$, only when the two col vectors are lin. indep