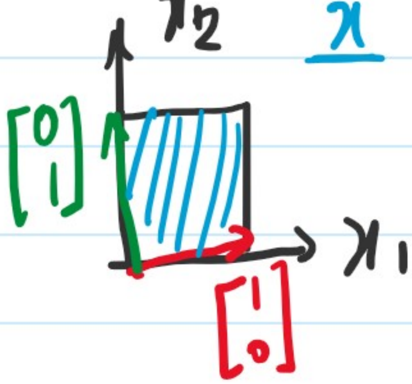
EECS 16A
Lecture 2A
July 6, 2020

Topics:
Determinant
Page Rank
Eigen value/vectors

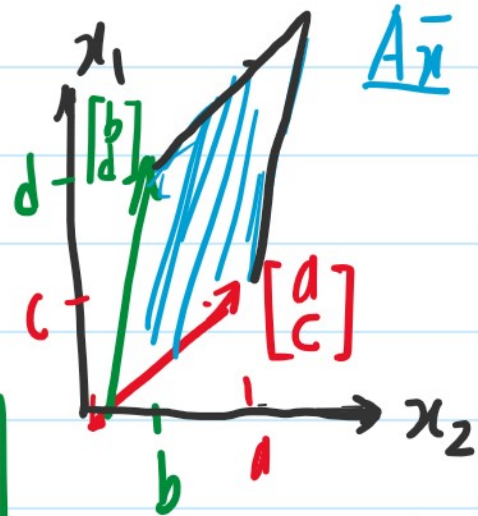
Announcements:
1) HW1B is due tonight
2) HW2A will be up today (due Wed)
3) Extra HW party today
4) HW2B will be mostly practice

Determinant: For a matrix $A \in \mathbb{R}^{n \times n}$, det(A): volume of n-dimensional parallelepiped spanned by the columns of A.
Change of volume caused by linear transformation A

Example: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$



$$\xrightarrow{Ax}$$



$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

Volume = 1

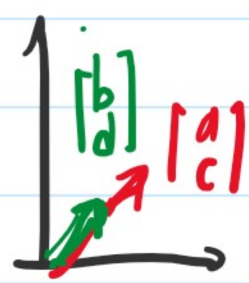
Volume = ad-bc

$$\text{Change of volume} = \frac{ad-bc}{1} = ad-bc$$

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc \rightarrow \text{2nd degree polynomial}$$

★ ad-bc is negative, if ad<bc
det(A) can be negative

★ det(A) = 0?



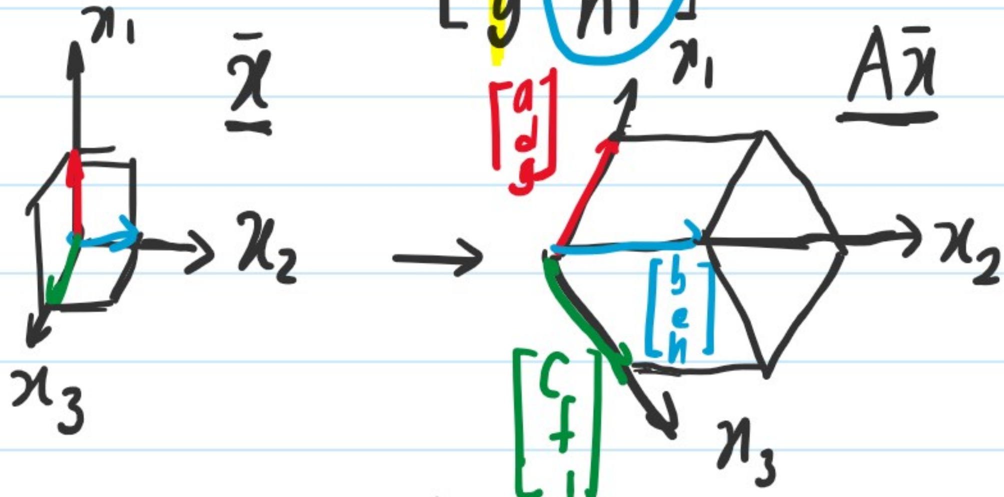
Vol = 0

⇔ Collapse of a dimension

⇔ non-invertible

⇔ Columns are lin. dependent

Example: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in \mathbb{R}^{3\times 3}$



$$\det(A) = a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

$= a(ei - fh) + \ldots\ldots$

= polynomial of degree 3

$A \in \mathbb{R}^{n\times n} \Rightarrow \det(A) \rightarrow$ polynomial of of n^{th} degree

Summary:

Equivalent statements for $A \in \mathbb{R}^{n \times n}$

Matrix is invertible

- Columns / rows are lin indep
- $A\bar{x}=b$ has a unique sol^n
- $\bar{b} \in C(A)$ for any $\bar{b}$
- Nullspace is trivial $= \{0\}$

  Nullity $= \dim(N(A)) = 0$
- Columnspace $\rightarrow \mathbb{R}^n$

  Rank $= \dim(C(A)) = n$

  $=$ # columns in A

  * Full rank matrix
- Determinant $\neq 0$

Eigen value & Eigen vectors:

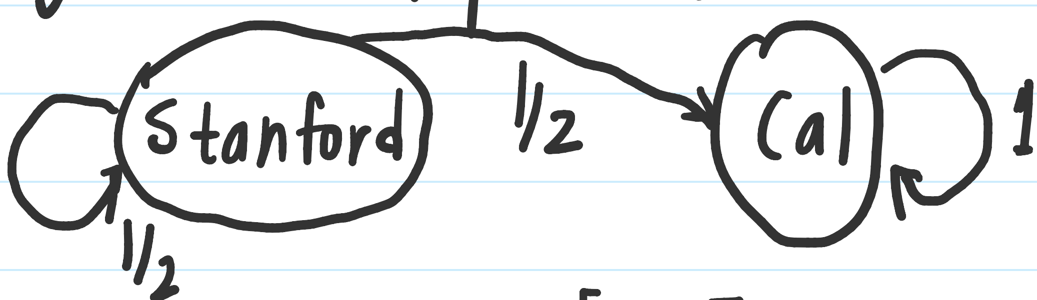
For $A \in \mathbb{R}^{n \times n}$, λ is an eigen value and $\bar{v}$ is the corresponding eigen vector if

$$A\bar{v} = \lambda \bar{v}$$

(λ is a scalar)

Page Rank: Ordering of webpages in search results.

Page rank example: Web traffic



Population, $\bar{x} = \begin{bmatrix} x_{st} \\ x_{cal} \end{bmatrix}$

Initial population, $\bar{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Linear transformation, $g = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$

$$\begin{array}{cc} \bar{1} & \bar{1} \end{array}$$

$$\bar{x}[n+1] = g\bar{x}[n]$$

$$\bar{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\bar{x}[1] = g\bar{x}[0] = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\bar{x}[2] = g\bar{x}[1] = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$$

$$\vdots$$

$$x[n] = \begin{bmatrix} 1/2^n \\ 1 - \frac{1}{2^n} \end{bmatrix}$$ → conservative system $x_{st} + x_{cal} = 1$

$$\bar{x}[\alpha] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ → steady state

$$\bar{x}_{steady} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$g\bar{x}_{steady} = \bar{x}_{steady}$$

$$\Rightarrow g\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \checkmark$$

$$\mathcal{G}\bar{x}_{steady} = \bar{x}_{steady} \qquad \Big| \; A\bar{v} = \lambda\bar{v}$$

$$\Rightarrow \mathcal{G}\bar{x}_{steady} = \bar{x}_{steady}$$

for $\lambda = 1$

★ $\bar{x}_{steady}$ is an eigenvector of $\mathcal{G}$ for $\lambda = 1$

★ System "might" reach steady state if $\lambda = 1$

Finding steady-state:

$$\mathcal{G}\bar{x}_{steady} = \bar{x}_{steady}$$

$$\Rightarrow \mathcal{G}\bar{x}_{steady} = I\bar{x}_{steady}$$

$$\Rightarrow (\mathcal{G} - I)\bar{x}_{steady} = \bar{0}$$

Solve for $\bar{x}_{steady}$

$$\bar{x}_{steady} \in N(\mathcal{G} - I)$$

$$[G - I \mid 0] \Rightarrow \left[\begin{array}{cc|c} \frac{1}{2}-1 & 0 & 0 \\ \frac{1}{2} & 1-1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{array}\right]$$

$$\rightarrow \begin{array}{c} \begin{array}{cc} x_{st} & x_{cal} \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

Basic var: $x_{st} = 0$

Free var : $x_{cal} = x_{cal}$

$$\bar{x}_{steady} = \begin{bmatrix} 0 \\ x_{cal} \end{bmatrix} = x_{cal} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Eigen vectors: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$

Eigen space: span$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

★ Show that eigenvectors corresponding to an eigenvalue λ form a subspace

p1) Closed under vec add

p2) " " scalar mult

p3) $\bar{0}$ is included

(P1 + P2) If $\bar{v}_x$, $\bar{v}_y$ are two eigenvectors of A for eigenvalue λ, then $\alpha\bar{v}_x + \beta\bar{v}_y$ should be another eigenvector for λ. α, β are scalars

Given: $A\bar{v}_x = \lambda\bar{v}_x$ (i) $\quad A\bar{v}_y = \lambda\bar{v}_y$

(i) $\times \alpha$ + (ii) $\times \beta$

$\Rightarrow$

Target: $A(\alpha\bar{v}_x + \beta\bar{v}_y) = \lambda(\alpha\bar{v}_x + \beta\bar{v}_y)$

(P3) $A\bar{0} = \lambda\bar{0}$

$\bar{0}$ is in the subspace.

Eigen space:

$E_\lambda = \{ \bar{v} : A\bar{v} = \lambda \bar{v} \}$ is a subspace called eigenspace

Finding eigenspaces:

$$A\bar{v} = \lambda\bar{v}$$
$$\Rightarrow A\bar{v} = \lambda I \bar{v}$$
$$\Rightarrow \underline{(A - \lambda I)}\,\bar{v} = 0$$

Solve for $\bar{v} \neq 0$

$A\bar{x} = \bar{0}$

$(A-\lambda I) \rightarrow$ Non-invertible
$\rightarrow$ determinant $= 0$

$\det(A-\lambda I) = 0$

$\Rightarrow$ polynomial of λ of degree $n = 0$

$\Rightarrow$ Solve for $\lambda_1, \lambda_2 \dots \lambda_n$

Step 1 : $\det(A-\lambda I)=0$

$\Rightarrow$ Solve for $\lambda_1 \cdots \lambda_n$

Step 2 : Nullspace of $(A-\lambda_1 I)$

$\rightarrow$ eigenspace for λ_1

$\vdots$

Nullspace of $(A-\lambda_n I)$

$\rightarrow$ eigenspace for λ_n

Example:

$$S=\begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

Eigen values: $\det(S-\lambda I)=0$

$$\Rightarrow \det\left(\begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}-\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)=0$$

$$\Rightarrow \det\left(\begin{bmatrix} 1/2-\lambda & 0 \\ 1/2 & 1-1 \end{bmatrix}\right)=0$$

$$\Rightarrow \left(\frac{1}{2}-\lambda\right)(1-\lambda)-0\times\frac{1}{2}=0$$

$$(\lambda-\tfrac{1}{2})(\lambda=1)=0$$

$$\Rightarrow \lambda_1 = 1$$
$$\lambda_2 = \tfrac{1}{2}$$

Eigen values

Eigen vectors:

$\lambda_1 = 1$: $E_{\lambda_1} = \text{Span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$ ← already solved

$\lambda_2 = \tfrac{1}{2}$: $g\bar{v}_2 = \lambda_2 \bar{v}_2$

$$\Rightarrow g\bar{v}_2 = \tfrac{1}{2}\bar{v}_2 = \tfrac{1}{2} I \bar{v}_2$$

$$\Rightarrow (g-\tfrac{1}{2}I)\bar{v}_2 = 0$$

Solve for $\bar{v}_2$

$$[g-\tfrac{1}{2}I \mid 0]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1/2-1/2 & 0 & 0 \\ 1/2 & 1-\tfrac{1}{2} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l}\text{Free var: } x_{cal} = x_{cal} \\ \text{Basic var: } x_{st} = -x_{cal}\end{array}\right\}$$

$$\bar{v}_2 = \begin{bmatrix} -x_{cal} \\ x_{cal} \end{bmatrix} = x_{cal}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

$$8 = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} : \quad \lambda_1 = 1, \; E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

$$\lambda_2 = \frac{1}{2}, \; E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

If $\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, are $\bar{v}_1$ & $\bar{v}_2$ linearly independent?

Any $x[0] \in \mathbb{R}^2$ can be expressed as a linear combination of $\bar{v}_1$ & $\bar{v}_2$

$$\text{span}\{\bar{v}_1, \bar{v}_2\} \Rightarrow \mathbb{R}^2$$

Eigen decomposition

Expressing a vector as a linear combination of eigen vectors. → Lecture 2B

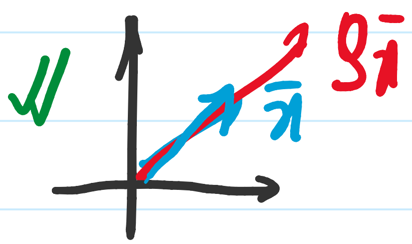
Visualization of eigen values/vectors

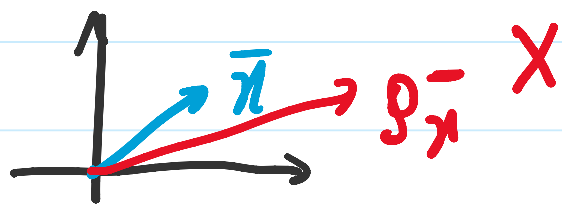
Ipython problem

Ex1: $S\bar{x} = \lambda\bar{x}$

$S\bar{x}$ is a scaled version of $\bar{x}$,

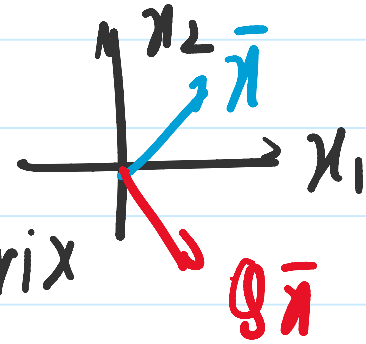
if $\bar{x}$ is an eigen vector of $\bar{x}$





Find eigen vectors

Ex 2 $\quad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



Reflection matrix

Find $\bar{x}$,
so that $S\bar{x}$ is going to be a
a scaled version of $\bar{x}$.

Finding eigenvectors visually.

→ possible answers $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda = 1$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \lambda = -1$

Ex 3 $\quad S = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix}$

Repeat for rotation matrix

→ Not possible for any real vectors.

→ Complex λ & $\bar{v}$ (Out of scope)