EECS 16A

Lecture 2B

July 7, 2020

Topics

Eigen values / vectors

Eigen decomposition

System stability

Announcements:

1) HW2B will be up today (One mandatory problem + practice problems)

2) HW2B solutions (practice problems) will be posted soon

3) HW2A is due tomorrow

4) Lecture 2C will be review

Page rank problem :

$$\vec{x} = \begin{bmatrix} x_{st} \\ x_{cal} \end{bmatrix} \quad Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

$$\lambda_1 = 1, \quad \bar{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{2}, \quad \bar{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

linearly independent

Assume, $\bar{x}[0] = \begin{bmatrix} \alpha \\ 1-\alpha \end{bmatrix}$, α is a scalar <1

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \bar{v}_1 - \alpha \bar{v}_2$$

Find $\bar{x}[t]$

$$\bar{x}[0] = \bar{v}_1 - \alpha \bar{v}_2$$

$$\bar{x}[1] = g\, x[0] = \underline{g \bar{v}_1} - \alpha \underline{g \bar{v}_2}$$

$$= \lambda_1 \bar{v}_1 - \alpha \lambda_2 \bar{v}_2$$

$g \bar{v}_1 = \lambda_1 v_1$

$g \bar{v}_2 = \lambda_2 \bar{v}_2$

$$\bar{x}[2] = g\, x[1]$$

$$= \lambda_1 \underbrace{g \bar{v}_1} - \alpha \lambda_2 \underbrace{g \bar{v}_2}$$

$$= \lambda_1^2 v_1 - \alpha \lambda_2^2 \bar{v}_2$$

$$\vdots$$

$$\bar{x}[t] = \lambda_1^t \bar{v}_1 - \alpha \lambda_2^t \bar{v}_2$$

$$\bar{x}[t] = \lambda_1^{\,t}\bar{v}_1 - \alpha\lambda_2^{\,t}\bar{v}_2$$
$$= \underbrace{1^t\bar{v}_1}_{\text{Steady}} - \underbrace{\alpha\left(\tfrac{1}{2}\right)^t\bar{v}_2}_{\text{Diminishing}}$$

$$\bar{x}[\infty] \approx \bar{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \leftarrow \text{steady state.}$$

Are eigen vectors for different λs always lin. indep?

Theorem: If $A \in \mathbb{R}^{n \times n}$ has n distinct eigen values $\lambda_1, \dots \lambda_n$, then corresponding eigenvectors $\{v_1, v_2 \dots v_n\}$ will form a basis for $\mathbb{R}^n$

Finding $\bar{x}[t]$ for any $x[0]$

Step 1: $\lambda_1, \ldots \lambda_n$: Distinct eigenvalues

Step 2: $\bar{v}_1, \ldots v_n$

Step 3: $\bar{x}[0] = c_1\bar{v}_1 + \cdots + c_n\bar{v}_n$

Step 4: $G^t \bar{x}[0] = ?$

Example: $G = \begin{bmatrix} 1 & 1 \\ 1/2 & 3/2 \end{bmatrix}$

Step 1: Eigen val: $\det(G - \lambda I) = 0$

$$\Rightarrow \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1/2 & 3/2-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \lambda^2 - \frac{5}{2}\lambda + 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - \frac{1}{2}\lambda + 1 = 0$$

$$\Rightarrow (\lambda - 2)\left(\lambda - \frac{1}{2}\right) = 0$$

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = 2, \quad \lambda_1 \neq \lambda_2$$

Step 2: Eigen vectors:

$\lambda_1 = \frac{1}{2}$, $\quad \mathcal{G}\bar{v}_1 = \frac{1}{2}\bar{v}_1 = \frac{1}{2}I\bar{v}_1$

$$\Rightarrow \left(\mathcal{G} - \frac{1}{2}I\right)\bar{v}_1 = 0$$

$$\left[\mathcal{G} - \frac{1}{2}I \,\middle|\, 0\right]$$

$\bar{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

$\lambda_2 = 2$: $\left[\mathcal{G} - 2I \,\middle|\, 0\right]$

$\bar{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

Finding time response $x[t]$.

<u>Method I</u>

$$x[1] = \mathcal{G}\bar{x}[0]$$

$$x[2] = \mathcal{G}\bar{x}[1] = \mathcal{G}^2\bar{x}[0]$$

$$\vdots$$

$$x[t] = \underbrace{\mathcal{G}^t}_{\text{Computational burden}}\bar{x}[0]$$

X

Method II : Eigen decomposition: Step 3

$$\bar{x}[0] = c_1\bar{v}_1 + c_2\bar{v}_2$$

Assume, $\bar{x}[0] = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ (Any $x[0] \in \mathbb{R}^2$ will do)

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \leftarrow \text{GE}$$

Step 4 : $\bar{x}[t]$

$$\bar{x}[0] = -1\cdot\bar{v}_1 + 1\cdot\bar{v}_2$$
$$= -\bar{v}_1 + \bar{v}_2$$
$$\bar{x}[1] = +g\,x[0]$$
$$= -g\bar{v}_1 + g\bar{v}_2$$
$$= -\lambda_1\bar{v}_1 + \lambda_2\bar{v}_2$$

$$\bar{x}[2] = -\lambda_1^{2}\bar{v}_1 + \lambda_2^{2}\bar{v}_2$$

$$\vdots$$

$$\bar{x}[t] = -\lambda_1^{t}\bar{v}_1 + \lambda_2^{t}\bar{v}_2$$

$$= \underbrace{-\left(\tfrac{1}{2}\right)^{t}\bar{v}_1}_{\text{Diminishes}} + \underbrace{2^{t}\bar{v}_2}_{\text{Explodes}}$$

Cases — State of system

1. $\bar{x}[0] = c_1\bar{v}_1 \rightarrow \bar{x}[t] = c_1\left(\tfrac{1}{2}\right)^{t}\bar{v}_1 \rightarrow 0$ Stable

II) $\bar{x}[0] = c_2\bar{v}_2 \rightarrow \bar{x}[t] = c_2 2^{t}\bar{v}_2 \rightarrow \infty$ Unstable

III) $\bar{x}[0] = c_1\bar{v}_1 + c_2\bar{v}_2 \rightarrow \bar{x}[t] = c_1\left(\tfrac{1}{2}\right)^{t}\bar{v}_1 + c_2 2^{t}\bar{v}_2 \rightarrow \infty$ Unstable

[python demo Ex1: I) $\bar{x}[0] = c_1 v_1 \rightarrow 0$

II) $\bar{x}[0] = c_2 v_2 \rightarrow \infty$

Example : $A \in \mathbb{R}^{3\times 3}$

I) $\lambda_1 = 2$, $\lambda_2 = \frac{1}{2}$, $\lambda_3 = 1$

II) $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$

$\lambda_1 \neq \lambda_2 \neq \lambda_3$

$\text{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} = \mathbb{R}^3$

III) $\bar{x}[0] = c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3$

GE: Solve for c_1, c_2, c_3

IV) $\bar{x}[t] = c_1\lambda_1^t\bar{v}_1 + c_2\lambda_2^t\bar{v}_2 + c_3\lambda_3^t\bar{v}_3$

$$= \underbrace{c_1 2^t \bar{v}_1}_{\text{Explodes}} + \underbrace{c_2\left(\tfrac{1}{2}\right)^t\bar{v}_2}_{\text{Diminishes}} + \underbrace{c_3 1^t \bar{v}_3}_{\text{Steady}}$$

Cases	Stability
$\bar{x}[0] = c_1\bar{v}_1$	Unstable $\rightarrow \infty$
$\bar{x}[0] = c_2\bar{v}_2$	Stable $\rightarrow 0$
$\bar{x}[0] = c_3\bar{v}_3$	Stable $\rightarrow$ Steady State $c_3\bar{v}_3$

iPython Ex 2

Case	Stability
$\bar{x}[0] = c_1\bar{v}_1 + c_2\bar{v}_2$	Unstable $\to \infty$
$\bar{x}[0] = c_2\bar{v}_2 + c_3\bar{v}_3$	Stable $\to c_3\bar{v}_3$

* $\lambda = 1$ does not guarantee steady state, depends on other λs.

→ System will be stable for any $x[0]$, when $\lambda_1, \lambda_2 \dots \lambda_n < 1$

What happens for a negative eigenval?

$\lambda_1 = -1$, $\quad c_1\lambda_1^t\bar{v}_1 \quad c_1(-1)^t\bar{v}_1$

$$= \begin{cases} c_1\bar{v}_1, & t \text{ is even} \\ -c_1\bar{v}_1, & t \text{ is odd} \end{cases}$$

→ Oscillating

* For $A \in \mathbb{R}^{2\times 2}$, if $\lambda_1 \neq \lambda_2$, then $\bar{v}_1$ & $\bar{v}_2$ are going to form a basis for $\mathbb{R}^2$

Proof

Given: $\underline{A\bar{v}_1 = \lambda_1 \bar{v}_1}$ (I) $\quad$ $\underline{A\bar{v}_2 = \lambda_2 \bar{v}_2}$ (II)

$\lambda_1 \neq \lambda_2$

Proof by contradiction: Assume $\bar{v}_1$ & $\bar{v}_2$ are linearly dependent

$\bar{v}_1 = \alpha \bar{v}_2$, $\alpha \neq 0$

$\Rightarrow A\bar{v}_1 = A\alpha\bar{v}_2$

$\Rightarrow \underline{A\bar{v}_1} = \alpha \underline{A\bar{v}_2}$

$\Rightarrow \lambda_1 \bar{v}_1 = \alpha \lambda_2 \bar{v}_2 \Rightarrow \lambda_1 \bar{v}_1 = \lambda_2 (\underline{\alpha \bar{v}_2})$

$\Rightarrow \lambda_1 \bar{v}_1 = \lambda_2 \bar{v}_1$

$\Rightarrow \boxed{\lambda_1 = \lambda_2}$ Contradiction

Target: $\{v_1, v_2\}$ form a basis for $\mathbb{R}^2$

i.e. $\bar{v}_1$ & $\bar{v}_2$ are linearly indep

$\underline{\lambda_1 = \lambda_2}$

$$\mathcal{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \det(\mathcal{G} - \lambda I) = 0$$

$$\Rightarrow (\lambda - 1)^2 = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = 1 .$$

Eigen vectors:

$$(\mathcal{G} - I)\bar{v} = 0$$

$$\Rightarrow [\mathcal{G} - I \mid 0] .$$

$$\Rightarrow \begin{array}{c} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cc} x_A & x_B \end{array} \end{array} \qquad \begin{array}{l} x_A = \text{free} \\ x_B = \text{free} \end{array}$$

$$\bar{v} = \begin{bmatrix} x_A \\ x_B \end{bmatrix} = x_A \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_B \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

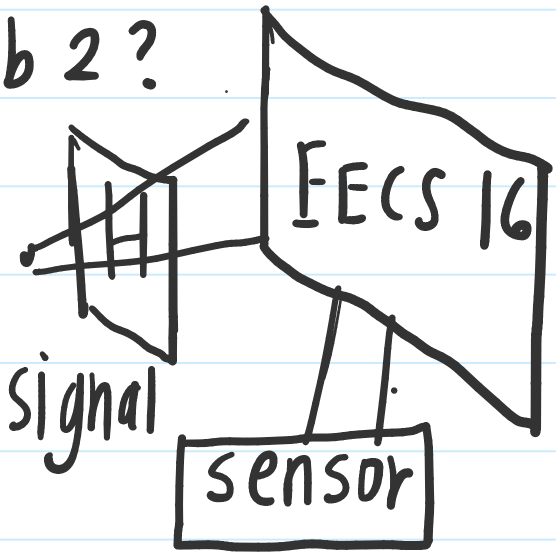
Eigen space: E_{λ_1, λ_2}

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow \mathbb{R}^2 \rightarrow \dim = 2$$

Imaging Lab

HW2B → Prob 2 ?



$\bar{i}$ = image

H = mask

$\bar{s}$ = sensor signal

$\bar{s} = H\bar{i}$

$\Rightarrow \bar{i} = H^{-1}\bar{s}$ ← Imaging 2 lab

Imaging 3 lab: $\bar{w}$ = noise

$\bar{s} = H\bar{i} + \bar{w}$

$\Rightarrow H^{-1}\bar{s} = \underbrace{\bar{i}}_{\text{need to retrieve}} + \boxed{H^{-1}w}$

need to minimize

★ To minimize $H^{-1}w$, H^{-1} should have small eigenvalues (<1)

$$\boxed{H\bar{v} = \lambda\bar{v}}$$

H: λ = eigen value
$\bar{v}$ = eigen vector

$$\Rightarrow H^{-1}H\bar{v} = H^{-1}\lambda\bar{v}$$
$$\Rightarrow I.\bar{v} = \lambda H^{-1}\bar{v}$$
$$\Rightarrow \bar{v} = \lambda H^{-1}\bar{v}$$
$$\Rightarrow \boxed{H^{-1}\bar{v} = \frac{1}{\lambda}\bar{v}}$$

H^{-1}: eigen val $= \frac{1}{\lambda}$
eigen vec $= \bar{v}$

$$\boxed{\begin{array}{l}\text{Maximize } \lambda \\ \text{Minimize } \frac{1}{\lambda}\end{array}}$$