EECS 16A
Lecture 2C
July 8, 2020

Topic
Review

Announcements:
- HW2B is up (Mostly practice for MT1)
- HW2B solutions will be up today
- Module 2 starts tomorrow. (Lec 2D)

Properties of an invertible matrix
$A \in \mathbb{R}^{n \times n}$
- Row/columns are lin indep
- $A\vec{x} = \vec{b}$ unique sol^n
- Columnspace of $A = C(A)$
  $= \text{Range}(A) = \mathbb{R}^n$
- $\text{Rank}(A) = \dim(C(A)) = n$
  ★ Full rank matrix
- Columns can form a basis for $\mathbb{R}^n$
- $\text{Nullity}(A) = \dim(N(A)) = 0$

- $\det(A) \neq 0$
- n non-zero eigen values
  (distinct/repeated)

☆ Zero eigen values

$$G\bar{v} = 0.\bar{v}$$
$$\Rightarrow \underbrace{G\bar{v} = 0}_{\text{Nontrivial Nullspace}}, \text{ for } \bar{v} \neq 0$$

Nontrivial Nullspace

$\therefore G$ is non-invertible

$\bar{v} \in$ Nullspace(G)

Eigen vector corresponding to $\lambda = 0$ will be in the nullspace

Ex: $G = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/4 & 3/4 & -1/4 \\ -1/4 & 1/4 & 1/4 \end{bmatrix}$

λ: $\det(G - \lambda I) = 0$

$$\Rightarrow (1-2\lambda)(1-4\lambda+4\lambda^2) - (1-2\lambda) = 0$$

$$\Rightarrow \lambda(1-2\lambda)(1-\lambda) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = \tfrac{1}{2}, \quad \lambda_3 = 1$$

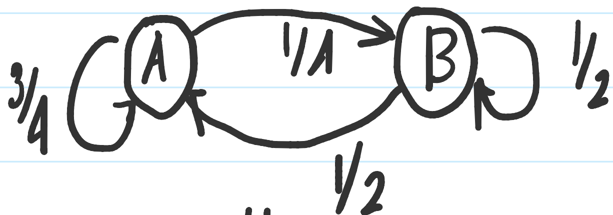
$\lambda_1 = 0$: $[\,G - \lambda_1 I \mid 0\,] \Rightarrow \bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = \frac{1}{2}$: $[\,G - \lambda_2 I \mid 0\,] \Rightarrow \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_3 = 1$: $[\,G - \lambda_3 I \mid 0\,] \Rightarrow \bar{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

lin indep

Page rank: web traffic



$\bar{x} = \begin{bmatrix} x_A \\ x_B \end{bmatrix}$

$$G = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$$

1) What $\bar{x}[0]$ makes this system unstable?

Eigen values: $g = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$

$\det(g - \lambda I) \Rightarrow \lambda_1 = 1, \lambda_2 = \frac{1}{4}$

* Eigenvectors corresponding to $\lambda > 1$ make system unstable

This system is stable for any value of $\bar{x}[0]$

* What will be steady state (if any) for $\bar{x}[0] = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$\lambda_1 = 1, \quad \bar{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \lambda_1 \neq \lambda_2$

$\lambda_2 = \frac{1}{4}, \quad \bar{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad \bar{v}_1$ & $\bar{v}_2$ are lin indep

$\bar{x}[0] = c_1 \bar{v}_1 + c_2 \bar{v}_2$

Long Process:

Eigen decomposition

$$\bar{x}[0] = c_1\bar{v}_1 + c_2\bar{v}_2$$

$$\Rightarrow \begin{bmatrix}3\\3\end{bmatrix} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\bar{x}[0] = 2\bar{v}_1 + \bar{v}_2$$

$$\begin{aligned}\bar{x}[t] &= P^t\,\bar{x}[0]\\ &= c_1\lambda_1^t\bar{v}_1 + c_2\lambda_2^t\bar{v}_2\\ &= \underbrace{2\cdot 1\cdot\bar{v}_1}_{\text{steady}} + \underbrace{1\cdot\left(\tfrac{1}{2}\right)^t\bar{v}_2}_{\text{Diminishes}}\end{aligned}$$

$$x[\infty] = 2\bar{v}_1 = 2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$$

$$= x_{steady}\ ?$$

$$g\,\bar{x}_{steady} = \bar{x}_{steady}$$

$$\begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$ *Verified

$$\bar{x}_{steady} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

For $t=\infty$, $x_A + x_B = 4+2 = 6$

For $t=0$, $x_A + x_B = 3+3 = 6$

Same

Conservative system

Shortcut: For conservative systems only

$$\bar{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow \text{Unrealistic population}$$

$\bar{x}[0]$ will not be a scaled version of $\bar{v}_2$

Any realistic $x[0]$, will include a component of $\bar{v}_1$

$$\bar{x}[t] = \underbrace{c_1}_{c_1 \neq 0}\bar{v}_1 + c_2\left(\tfrac{1}{2}\right)^t \bar{v}_2$$

$x[\infty] = c_1 \bar{v}_1 \rightarrow$ Steady state

$$\bar{x}[0] = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad x_A + x_B = 6$$

$$\bar{x}_{steady} = c_1 \bar{v}_1 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ c_1 \end{bmatrix}$$

$$x_A + x_B = 6$$
$$\Rightarrow 2c_1 + c_1 = 6 \Rightarrow c_1 = 2$$

Steady state, $\bar{x}_{steady}$

$$= 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Summary:

$$\bar{x}[0] = c_1 . 1^t . \bar{v}_1 + c_2 \left(\tfrac{1}{2}\right)^t \bar{v}_2$$
$$= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \left(\tfrac{1}{2}\right)^t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\bar{x}[0]$	$\bar{x}[\infty]$
$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\bar{v}_1$	$c_1 = 2,\ c_2 = 0$ $\bar{x}[\infty] = 2\bar{v}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$
$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \bar{v}_2$	$c_1 = 0,\ c_2 = 1$ $\bar{x}[\infty] = 0$
$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 2\bar{v}_1 + \bar{v}_2$	$c_1 = 2,\ c_2 = 1$ $x[\infty] = 2\bar{v}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Page ranking

$\bar{x}_{steady} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ i) A $x_A > x_B$

ii) B

Ex: Eigen vector/value

$g = \begin{bmatrix} 1 & 1 \\ 1/2 & 3/2 \end{bmatrix}$ $\lambda_1 = \frac{1}{2}$ $\lambda_2 = \underbrace{2}$ >1

$\bar{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\bar{v}_2 = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

* What $\bar{x}[0]$ will make this system unstable?

I) $\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\bar{v}_2 \rightarrow x[\alpha] = 3 \cdot \underbrace{2^t}\,\bar{v}_2$

Unstable

Stable II) $\begin{bmatrix} 4 \\ -2 \end{bmatrix} = -2\bar{v}_1 \rightarrow x[\alpha] = -2\left(\frac{1}{2}\right)^t \bar{v}_1 \rightarrow 0$

Stable

III) $\begin{bmatrix} 0 \\ 3 \end{bmatrix} = c_1\bar{v}_1 + c_2\underbrace{\bar{v}_2}, \; c_1 \neq 0, c_2 \neq 0$

$\rightarrow x[\alpha] \rightarrow \infty$ Unstable

IV) $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = c_1\bar{v}_1 + c_2\underbrace{\bar{v}_2}$ $c_1 \neq 0, c_2 \neq 0$

$\rightarrow x[\alpha] \rightarrow \infty$

Unstable

Having $\bar{v}_2$ in $\bar{x}[0]$ will make system unstable

★ Eigen vector/value summary for $A \in \mathbb{R}^{n \times n}$

i) For n distinct eigen values, $\bar{v}_1, \bar{v}_2 \dots \bar{v}_n$ will be lin. indep.
span$\{\bar{v}_1, \bar{v}_2, \dots \bar{v}_n\} = \mathbb{R}^n$

ii) For n distinct eigenvalues
$\bar{x}[0] = c_1\bar{v}_1 + \dots + c_n\bar{v}_n$ for $\bar{x}[0] \in \mathbb{R}^n$

iii) $Q^t \bar{x}[0] = c_1\lambda_1^t\bar{v}_1 + c_2\lambda_2^t\bar{v}_2 + \dots + c_n\lambda_n^t\bar{v}_n$

v) $Q\bar{x}_{steady} = \bar{x}_{steady}$

vi) A system with all eigenvalues ≤ 1 will always be stable.

vii) For a non-zero steady state, $\lambda = 1$ is required.

viii) Negative eigenvalues $\rightarrow x[t]$ oscillates.

Columnspace :

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 3 & 0 \\ 1 & 1 & 3 & 2 \end{bmatrix}$$

Which option represents columnspace?

i) $\text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\3\\3\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix}\right\}$ Vectors not basis

→ Yes / Definition of C(A)

ii) $\text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}\right\}$ → Yes Basis vectors

lin indep

How many lin. indep. columns in A?

$$[A|0] \Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 1 & 1 & 3 & 2 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3/2 & 2 & 0 \\ 0 & 1 & 3/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

\# lin indep columns = 2

Rank(A) = 2

iii) $\text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \underbrace{\begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}} \right\} \rightarrow$ Yes

Basis vectors

lin indep

iv) $\text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \underbrace{\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}} \right\} \rightarrow$ No

Not basis vectors

lin dep

Nullspace

Free var: $\underbrace{x_3 = x_3}$ $\underbrace{x_4 = x_4}$

Basic: $x_1 = -\frac{3}{2}x_3 - 2x_4$

$x_2 = -\frac{3}{2}x_2$

$$\bar{x} = \begin{bmatrix} -3/2 \\ -3/2 \\ 1 \\ 0 \end{bmatrix} \underline{x_3} + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \underline{x_4}$$ $\in$ Nullspace

lin indep?

Yes

Which space is the same as nullspace(A)?

i) $\mathrm{span}\left\{\begin{bmatrix}3/2\\3/2\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\\-1\end{bmatrix}\right\}$ Yes $\rightarrow$ 2D plane

ii) $\underbrace{\mathrm{span}\left\{\begin{bmatrix}-3/2\\-3/2\\1\\0\end{bmatrix}\right\}} + \underbrace{\mathrm{span}\left\{\begin{bmatrix}-2\\0\\0\\1\end{bmatrix}\right\}}$ No

$\rightarrow$ 2 lines

Doesn't represent the 2D plane

Ex 1: $A \in \mathbb{R}^{3\times 3}$, $B \in \mathbb{R}^{3\times 3}$, $\mathrm{Rank}(A)=3$, $\mathrm{Rank}(B)=2$, Which options are invertible?

- i) A Yes
- ii) B No
- iii) AB No
- iv) BA No

iii) $x \in \mathbb{R}^3 \rightarrow$ 3D volume

$B\bar{x} \rightarrow$ 2D plane

$\mathrm{Nullity}(B)=1$ $\mathrm{Rank}(B)=2$

$A(B\bar{x}) \rightarrow$ 2D plane

3D $\rightarrow$ 2D : irreversible process

iv) $\bar{x} \in \mathbb{R}^3 \rightarrow$ 3D volume

$A\bar{x} \Rightarrow \mathbb{R}^3 \rightarrow$ 3D volume

$B(A\bar{x}) \Rightarrow$ 2D plane

3D volume $\rightarrow$ 2D plane

Nullspace is not trivial

$BA \Rightarrow$ non-invertible

Linear tranform-ations can collapse dims but cannot create dims. Dimension loss $\rightarrow$ irreversible process

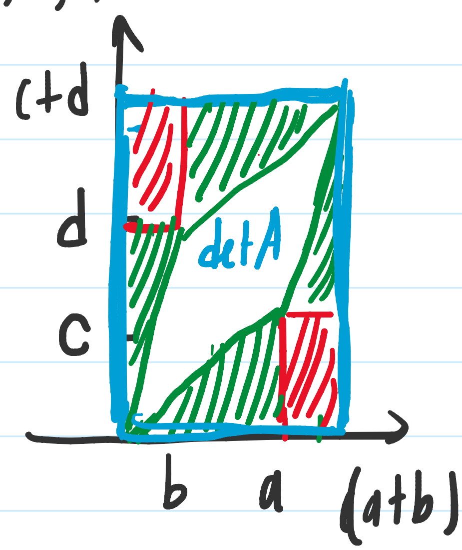
* $M = ABCD \quad M, A, B, C, D \in \mathbb{R}^{n \times n}$

if M is invertible, A, B, C, D must be invertible.

Determinant:

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

Proof:



$\det(A) = \square - \blacksquare - \blacktriangle$

$= (a+b)(c+d)$

$- bc - bc$

$- \frac{1}{2}ac - \frac{1}{2}ac - \frac{1}{2}bd - \frac{1}{2}bd$

Transformation matrix

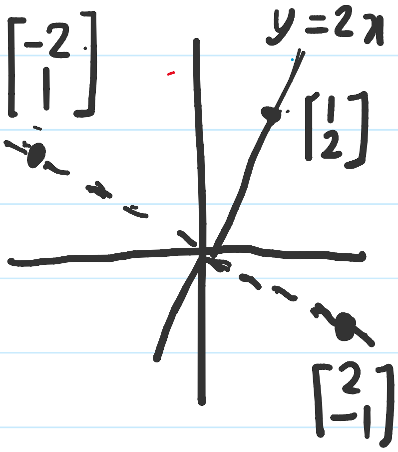
Scaling: $\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$

Reflection: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ → across x-axis

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ → across y-axis

$\begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$ → Reflection across some line

Reflection across $y=2x$



Assume $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Solve for a, b, c, d