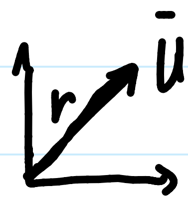
EECS 16A
Module 3
Lecture 2 (6A)

Topics
Classification of signal
Estimation of signal delay

* HW6A will be up today
  Problems covering lec 6A
* Last lecture : Norm, inner product
  Classification

Norm: 

r = Norm of $\bar{u}$
$= \|u\| = \sqrt{u_1^2 + \cdots + u_n^2}$

Inner product:
Inner product of $\bar{u}$ & $\bar{v}$ $= \langle \bar{u}, \bar{v} \rangle$
$= u_1 v_1 + \cdots + u_n v_n$
$= \bar{u}^T \bar{v}$ T → transpose

* $\langle \bar{u}, \bar{v} \rangle = \langle \bar{v}, \bar{u} \rangle$
* $\langle \bar{u}, \bar{u} \rangle = u_1^2 + \cdots u_n^2 = \|u\|^2$

Database:

$\bar{s}_1 = [1 \;\; 1 \;\; -1 \;\; -1 \;\; 1 \;\; 1]^T$ $\quad \|\bar{s}_1\| = \sqrt{6}$

$\bar{s}_2 = [1 \;\; -1 \;\; -1 \;\; 1 \;\; 1 \;\; 1]^T$ $\quad \|\bar{s}_2\| = \sqrt{6}$

Received.

$\bar{r} = [1 \;\; 1 \;\; -1 \;\; -1 \;\; 1 \;\; 1]^T = \bar{s}_1$

Method 1: $\bar{e}_1 = \bar{r} - \bar{s}_1$, $\|\bar{e}_1\| = 0$ minimum

$\bar{e}_2 = \bar{r} - \bar{s}_2$, $\|\bar{e}_2\| > 0$

Smaller $\|e\|$ → better match

Method 2:

$\langle \bar{r}, \bar{s}_1 \rangle = 1+1+1+1+1+1 = 6$

$= \|\bar{s}_1\|^2$

$\langle \bar{r}, \bar{s}_2 \rangle = 1-1+1-1+1+1 = 2 < 6$

Bigger inner product → better match.

Relationship between inner product & error

$$\text{error, } \bar{e} = \bar{r} - \bar{s}$$

$$\underline{\|e\|^2} = \langle \bar{e}, \bar{e} \rangle = \bar{e}^T \bar{e}$$
$$= (\bar{r} - \bar{s})^T (\bar{r} - \bar{s})$$
$$= (\bar{r}^T - \bar{s}^T)(\bar{r} - \bar{s})$$
$$= \bar{r}^T \bar{r} + \bar{s}^T \bar{s} - s^T \bar{r} - \bar{r}^T \bar{s}$$
$$= \|\bar{r}\|^2 + \|\bar{s}\|^2 - \langle \bar{s}, \bar{r} \rangle - \langle \bar{r}, \bar{s} \rangle$$
$$= \|\bar{r}\|^2 + \|\bar{s}\|^2 - \underline{2\langle \bar{r}, \bar{s} \rangle}$$

$(a+b)^T = a^T + b^T$ → Lec 5D

When $\langle \bar{r}, \bar{s} \rangle \to \max$

$\|\bar{e}\| \to \min$

Ex: $\bar{s}_1 = [1 \;\; 1 \;\; -1 \;\; -1 \;\; 1 \;\; 1]^T$ $\quad \|\bar{s}_1\| = \sqrt{6}$

$\bar{s}_2 = [4 \;\; -4 \;\; -4 \;\; 4 \;\; 4 \;\; 4]^T$ $\quad \|\bar{s}_2\| = 4\sqrt{6}$

$\bar{r} = [1 \;\; 1 \;\; -1 \;\; -1 \;\; 1 \;\; 1]^T = \bar{s}_1$

$\langle \bar{r}, \bar{s}_1 \rangle = 6 = \|\bar{s}_1\|^2$

$\langle \bar{r}, \bar{s}_2 \rangle = 4 - 4 + 4 - 4 + 4 + 4 = 8 > 6$

Prob: $\bar{r}$ matches $\bar{s}_1$, but $\langle \bar{r}, \bar{s}_2 \rangle$ is larger.

solution:

Design database signals with the same norm. $\|s_1\| = \|s_2\|$

HW: shazam problem

sending two signals simultaneously:

$$\bar{r} = \bar{s}_1 + \bar{s}_2 + \bar{n} \qquad \bar{n} = \text{noise}$$

$$\|\bar{s}_1\| = \|\bar{s}_2\|$$

$$\begin{aligned}\langle \bar{r}, \bar{s}_1 \rangle &= \langle \bar{s}_1 + \bar{s}_2 + \bar{n}, \bar{s}_1 \rangle \\ &= \langle \bar{s}_1, \bar{s}_1 \rangle + \langle \bar{s}_2, \bar{s}_1 \rangle + \langle \bar{n}, \bar{s}_1 \rangle \\ &= \|\bar{s}_1\|^2 + \underbrace{\langle \bar{s}_2, \bar{s}_1 \rangle + \langle \bar{n}, \bar{s}_1 \rangle}_{\text{Interference}}\end{aligned}$$

$$\langle \bar{s}_2, \bar{s}_1 \rangle = \|\bar{s}\| \cdot \|\bar{s}_1\| \underbrace{\cos(\theta_2 - \theta_1)}_{\text{Minimize}} \rightarrow \text{lec 5D}$$

Choose $\theta_2 - \theta_1 = 90°$

so that $\langle \bar{s}_2, \bar{s}_1 \rangle = 0$

i.e. make $\bar{s}_1$ & $\bar{s}_2$ orthogonal

$\langle \bar{n}, \bar{s}_1 \rangle =$ small if n is random noise

$$\langle \bar{r}, \bar{s}_1 \rangle \approx \|\bar{s}_1\|^2$$

$$\langle \bar{r}, \bar{s}_2 \rangle = \|\bar{s}_2\|^2 + \underbrace{\langle \bar{s}_2, \bar{s}_1 \rangle}_{0} + \underbrace{\langle \bar{n}, \bar{s}_2 \rangle}_{\text{small}}$$
$$\approx \|\bar{s}_2\|^2$$

Requirements for database:

* $\|\bar{s}_1\| = \|\bar{s}_2\|$
* $\bar{s}_1$ & $\bar{s}_2$ are orthogonal

$$\bar{s}_1 = [1 \ \ 1 \ \ -1 \ \ -1 \ \ 1 \ \ 1]^T$$
$$\bar{s}_2 = [1 \ \ -1 \ \ -1 \ \ 1 \ \ -1 \ \ 1]^T$$

$$\langle \bar{s}_1, \bar{s}_2 \rangle = 0$$

$$\|s_1\| = \|s_2\| = 6$$

If $\bar{r} = [1 \ \ -1 \ \ -1 \ \ 1 \ \ -1 \ \ 1] = \bar{s}_2$

$$\langle \bar{r}, \bar{s}_1 \rangle = \langle \bar{s}_2, \bar{s}_1 \rangle = 0$$
$$\langle \bar{r}, \bar{s}_2 \rangle = \langle \bar{s}_2, \bar{s}_2 \rangle = \|\bar{s}_2\|^2 = 6$$

Case I

$r = \bar{s}_1 + \bar{s}_2$

$\langle \bar{r}, \bar{s}_1 \rangle = 6$ ✓

$\langle \bar{r}, \bar{s}_2 \rangle = 6$ ✓

case II

$\bar{r} = \bar{s}_1$

$\langle \bar{r}, \bar{s}_1 \rangle = 6$ ✓

$\langle \bar{r}, \bar{s}_2 \rangle = 0$

case III

$\bar{r} = \bar{s}_2$

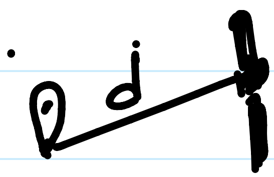
$\langle \bar{r}, \bar{s}_1 \rangle = 0$

$\langle \bar{r}, \bar{s}_2 \rangle = 6$ ✓

Toy GPS problem:

$s_1 = [1 \quad 1 \quad -1 \quad -1 \quad 1 \quad 1]^T$ $\quad \|\bar{s}_1\| = \|\bar{s}_2\|$

$s_2 = [1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1]^T$ $\quad \langle \bar{s}_1, \bar{s}_2 \rangle = 0$

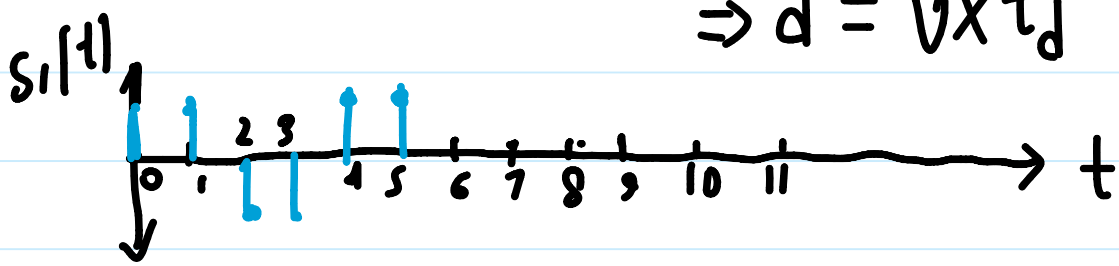


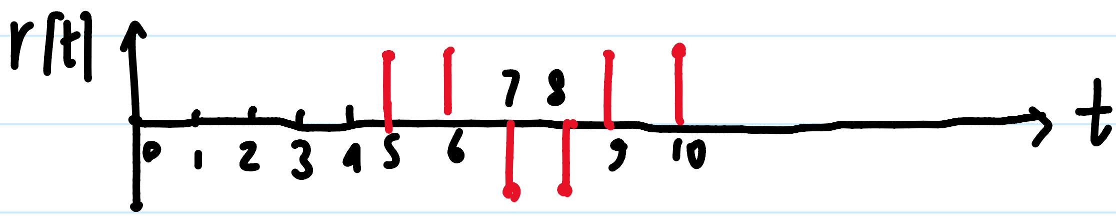
Distance = d

Signal velocity = v

Signal delay, $t_d = d/v$

$\Rightarrow d = v \times t_d$





From the plot

$s_1[0] = 1 = r[5]$

$s_1[1] = 1 = r[6]$

$s_1[2] = -1 = r[7]$

$s_1[3] = -1 = r[8]$

$s_1[4] = 1 = r[9]$

$s_1[5] = 1 = r[10]$

$r[t] = s_1[t-5]$

Inner product:

$\bar{r}[t] = [0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ -1 \ -1 \ 1 \ 1]$

$\bar{s}_1[t] = [1 \ 1 \ -1 \ -1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0]$

$\langle \bar{r}[t], \bar{s}_1[t] \rangle = 1 \neq 6$ ie. $\|\bar{s}_1\|^2$

$\bar{r}[t] = [0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ -1 \ -1 \ 1 \ 1]$

$\bar{s}_1[t-5] = [0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ -1 \ -1 \ 1 \ 1]$

$\langle \bar{r}[t], \bar{s}_1[t-5] \rangle = 6 = \|\bar{s}_1\|^2$

How do we know delay is 5

Find $\langle r[t], \bar{s}_1[t-k] \rangle$ for varying k

where k is the time shift of $\bar{s}_1[k]$

Powerpoint demo of sliding inner product

$$\langle \bar{r}[t], \bar{s}_1[t-k] \rangle$$

$$= \sum_{i=-\infty}^{\infty} r[i]\, \bar{s}_1[i-k]$$ → Generic expression for $-\infty$ to ∞

= cross-correlation of $\bar{s}_1$ w.r.t. $\bar{r}$

$= \text{corr}_r(s_1)[k]$

- k → time shift of $\bar{s}_1$
- s_1 → Shifting signal
- r → Stationary signal

$$\boxed{\begin{aligned} \text{corr}_r(s_1)[k] &= \sum_{i=-\infty}^{\infty} r[i]\, s_1[i-k] \\ &= \langle \bar{r}[t]\, \bar{s}[t-k] \rangle \end{aligned}}$$

Dis 6A, 6B: mech prob