EECS 16A
August 6, 2020
Lecture 6D
Grace Kuo

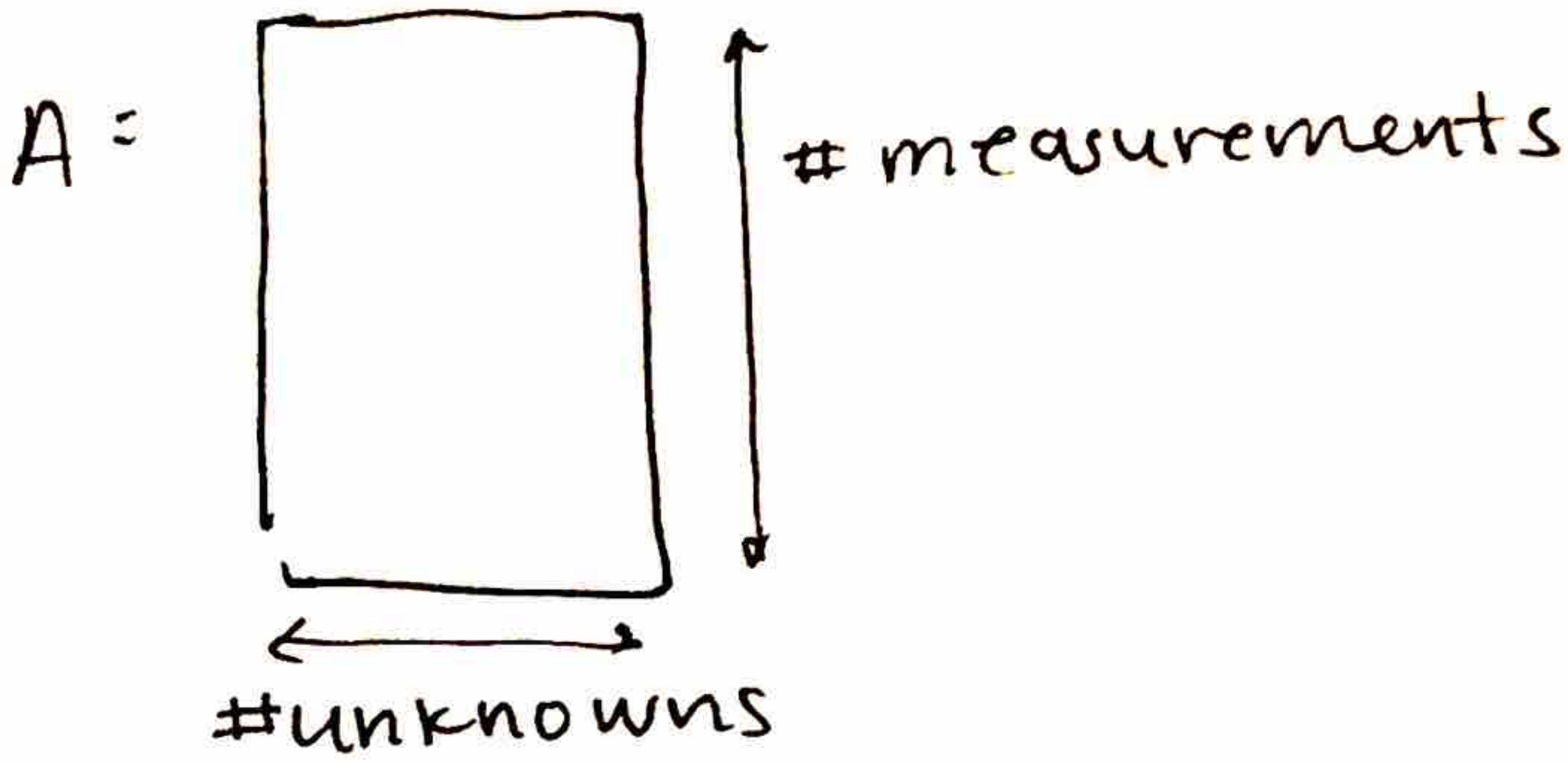
Today:

- Finish least squares
  - ↳ when does L.S. fail?
- Intro to OMP (orthogonal matching pursuit)

Yesterday: Least squares

$A\vec{x} = \vec{b}$ has no solution

because A is a "tall"



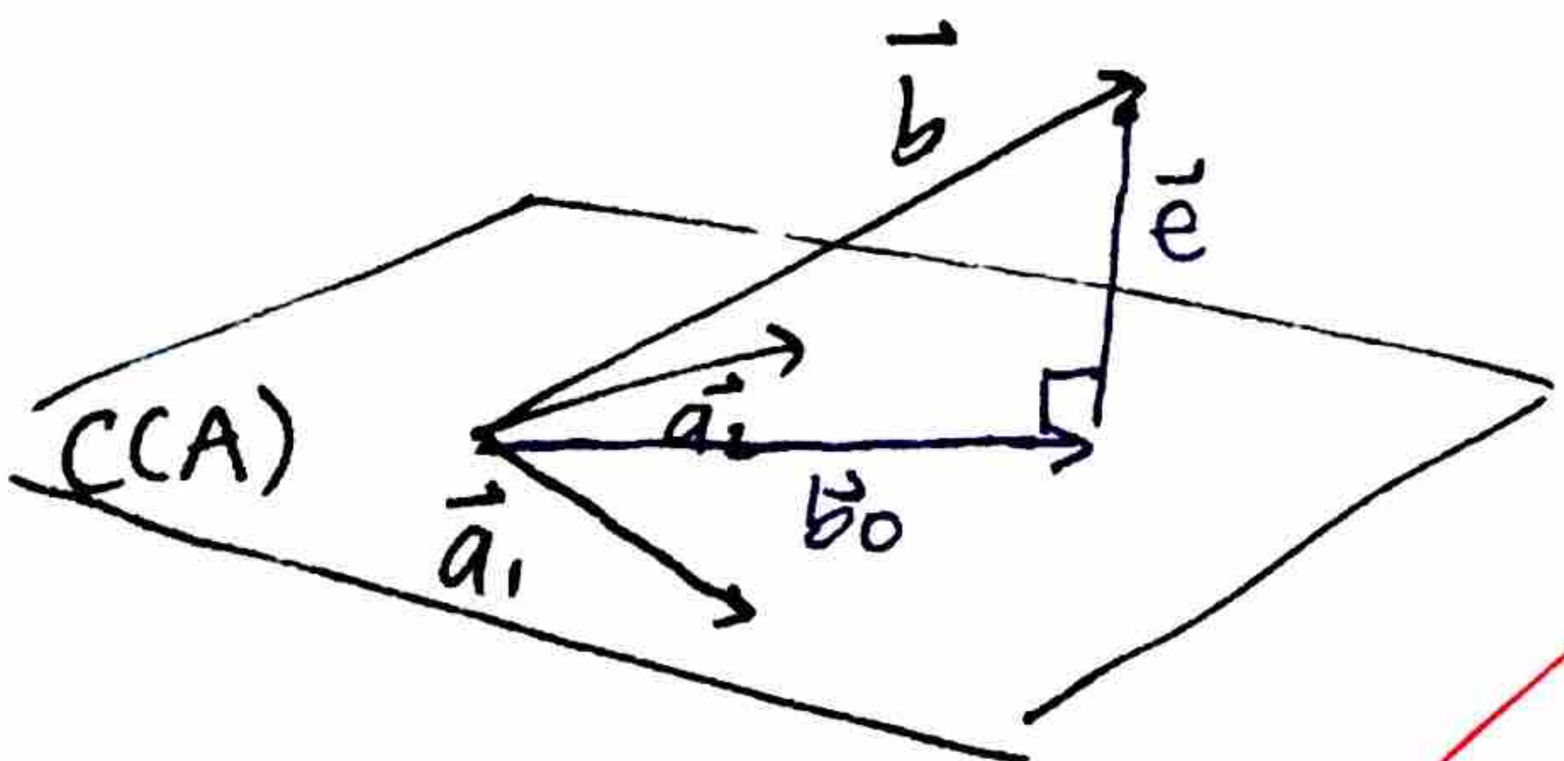
instead modify equation and solve

$$A\hat{x} = \vec{b}_0$$

best estimate of x

① $\vec{b}_0 \in C(A)$

② $\|\vec{b}_0 - \vec{b}\|$ is mimimized



$$\hat{x} = \boxed{(A^TA)^{-1}} A^T\vec{b}$$

what if A^TA isn't invertible?

$$\vec{b}_0 = A\hat{x} = A(A^TA)^{-1}A^T\vec{b}$$

$$\vec{e} = \vec{b} - \vec{b}_0$$

$\|\vec{e}\|^2 \leftarrow$ metric of how close $\vec{b}$ and $\vec{b}_0$ are "How good a fit it is"

When is $A^T A$ invertible?

Recall from mod. 1

$A^T A$ is invertible iff $N(A^T A)$ is trivial

" " is not invertible " " is non-trivial

$\boxed{\text{THM}}$ $N(A^T A) = N(A)$

Nullspace of $A^T A$ = Nullspace of A

what does this thm. tell us?

columns of A are lin. independent $\Leftrightarrow$ $N(A)$ is trivial $\Leftrightarrow$ $N(A^T A)$ is trivial $\Leftrightarrow$ $A^T A$ is invertible $\Leftrightarrow$ L.S. works!

columns of A are dependent $\longleftrightarrow$ L.S. does not work!! :(

Before Proving....

Transposes:

$A \in \mathbb{R}^{m \times n}$ $\quad B \in \mathbb{R}^{n \times p}$ $\quad C \in \mathbb{R}^{m \times p}$

$$C = AB$$
$$C^T = (AB)^T$$
$$\underset{p \times m}{C^T} = \underset{p \times n}{B^T} \; \underset{n \times m}{A^T}$$

$$\left.\underset{n \times m}{A^T}\;\underset{p \times n}{B^T}\right\}$$ can't even multiply

Try it: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Norms:

$\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$ ✓ property of norms

$$(\|\vec{x}\|)^2 = \left(\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}\right)^2 = (0)^2$$

$$\underbrace{x_1^2}_{x_1^2 \geq 0} + \underbrace{x_2^2}_{x_2^2 \geq 0} + \ldots + x_n^2 = 0$$

only sum to 0 if $x_1^2 = 0$, $x_2^2 = 0$, ...

$\hookrightarrow x_1 = 0$, $x_2 = 0$, ...

$\vec{x} = \vec{0}$

$\boxed{\text{THM}}$ $N(A^TA) = N(A)$

$\boxed{\text{proof}}$ Need to prove 2 things

1. if $\vec{v} \in N(A)$ then $\vec{v} \in N(A^TA)$
2. if $\vec{w} \in N(A^TA)$ then $\vec{w} \in N(A)$

1. Given: $\vec{v} \in N(A)$

$A\vec{v} = \vec{0}$

$A^TA\vec{v} = A^T\vec{0}$ left multiplied by A^T

$A^TA\vec{v} = \vec{0}$

Want to show: $\vec{v} \in N(A^TA)$ ✓ yay!

2 Given: $\vec{w} \in N(A^T A)$

$$A^T A \vec{w} = \vec{0}$$

$$w^T A^T A \vec{w} = \vec{w}^T \vec{0} = 0$$

$$\vec{w}^T A^T A \vec{w} = 0$$

$$(A\vec{w})^T (A\vec{w}) = 0$$

$$\|A\vec{w}\|^2 = 0^2 = 0$$

$$A\vec{w} = \vec{0} \iff \|A\vec{w}\| = 0$$

want to show: $\vec{w} \in N(A)$

$$\|\vec{x}\|^2 = \vec{x}^T \vec{x}$$

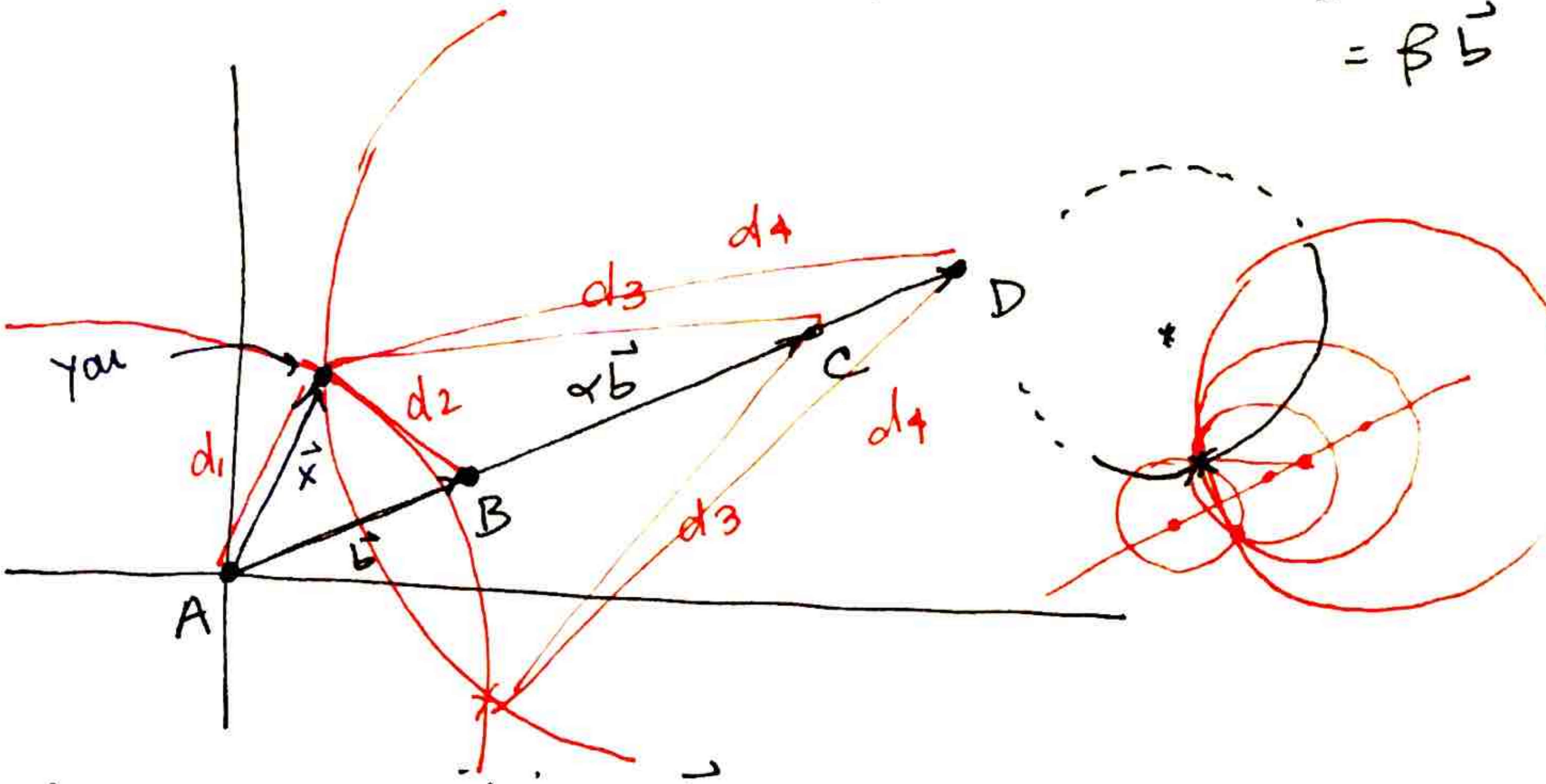
summary:
proved that we can look at the columns of A to determine if $(A^T A)^{-1}$ exists

Example when least squares fails

Trilateration

3 beacons (A, B, C)

$\vec{a} = \vec{0} \quad \vec{b} = \vec{b} \quad \vec{c} = \alpha\vec{b} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \vec{d} = \begin{bmatrix} \beta b_1 \\ \beta b_2 \end{bmatrix} = \beta\vec{b}$



Solve for our position, $\vec{x}$:

$$\underbrace{2\begin{bmatrix} \vec{b}^T - \vec{a}^T \\ \vec{c}^T - \vec{a}^T \end{bmatrix}}_{A} \vec{x} = \underbrace{\begin{bmatrix} d_1^2 - d_2^2 - \|\vec{a}\|^2 + \|\vec{b}\|^2 \\ d_1^2 - d_3^2 - \|\vec{a}\|^2 + \|\vec{c}\|^2 \end{bmatrix}}_{b}$$

$$A = \begin{bmatrix} 2\vec{b}^T \\ 2\alpha\vec{b}^T \end{bmatrix} = \begin{bmatrix} 2b_1 & 2b_2 \\ 2\alpha b_1 & 2\alpha b_2 \end{bmatrix}$$

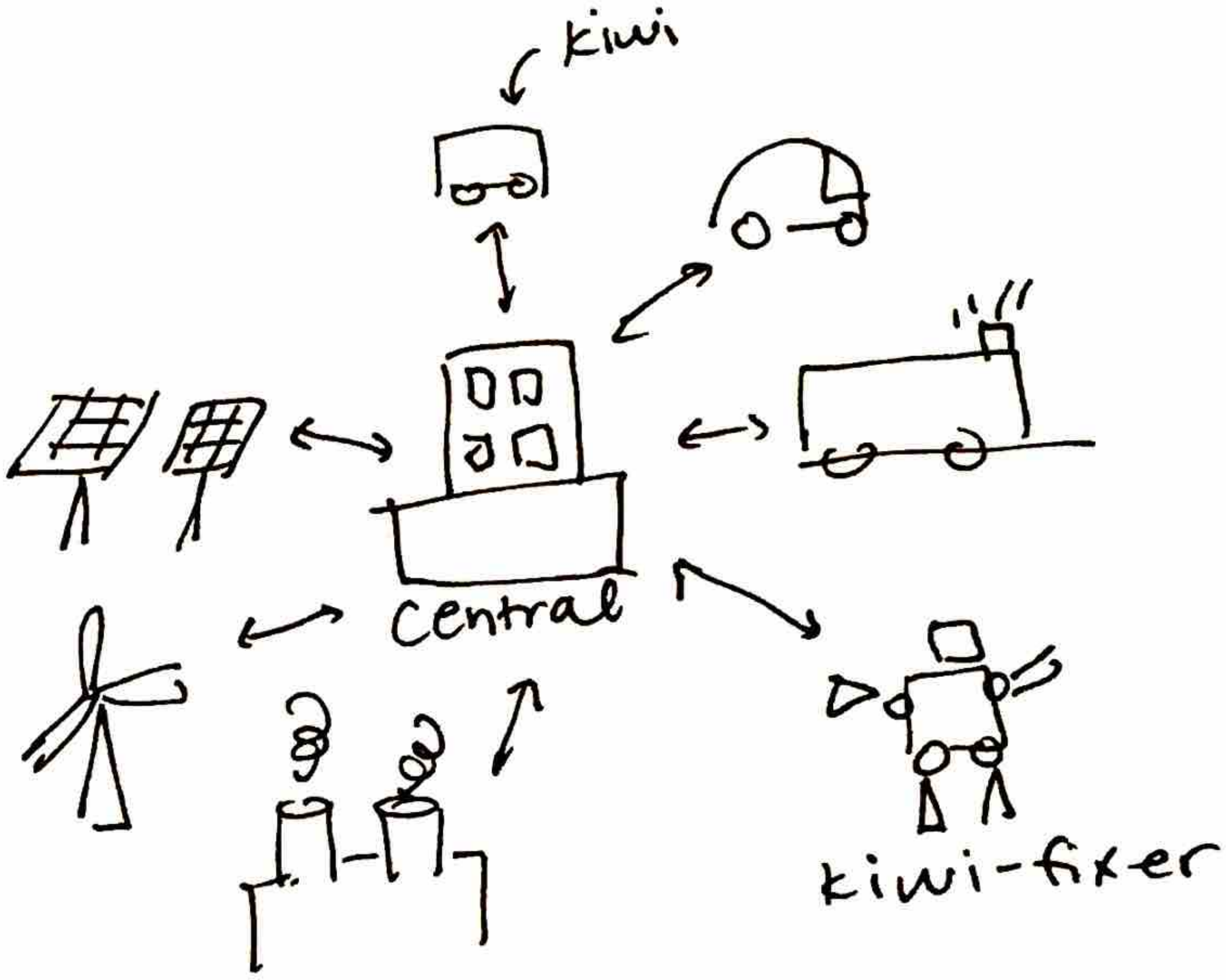
dependent columns !!

w/ 4 beacons:

$$A = 2\begin{bmatrix} \vec{b}^T - \vec{a}^T \\ \vec{c}^T - \vec{a}^T \\ \vec{d}^T - \vec{a}^T \end{bmatrix} = \begin{bmatrix} 2\vec{b}^T \\ 2\vec{c}^T \\ 2\vec{d}^T \end{bmatrix} = \begin{bmatrix} 2b_1 & 2b_2 \\ 2\alpha b_1 & 2\alpha b_2 \\ 2\beta b_1 & 2\beta b_2 \end{bmatrix}$$

New Topic: Orthogonal Matching Pursuit (OMP)

Application: Smart city

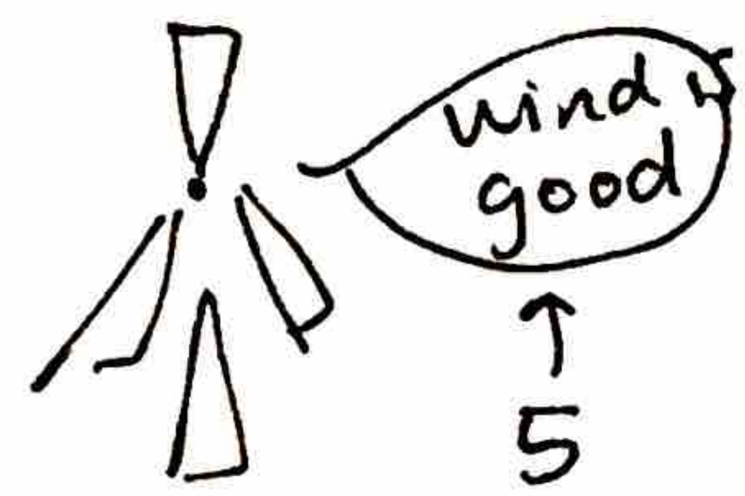


- lots of devices say we have 10,000 devices

- Each device has a signature $\vec{s}_i$ and if the device wants to send a message it sends

$$\alpha \vec{s}_i$$

α ← message, $\vec{s}_i$ ← signature



5 ... send $5\vec{s}_w$

$\vec{s}_w = \underbrace{[1\ -1\ 1\ -1 \ldots]}_{\text{Gold codes}}$

- Most devices don't have anything to say. They send $0\vec{s}_i$

... nothing to say sends $0\vec{s}_k$

- at central

$$\vec{r} = \alpha_1 \vec{s}_1 + \alpha_2 \vec{s}_2 + \ldots + \alpha_{10,000} \vec{s}_{10000}$$

$\vec{r}$ ← received signal

want to know:
1. which devices are transmitting
2. what are they saying? (α?)

write as a matrix equation:

$$\underbrace{\begin{bmatrix} | & | & & | \\ \vec{s}_1 & \vec{s}_2 & \cdots\cdots & \vec{s}_{10,000} \\ | & | & & | \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{10000} \end{bmatrix}}_{\vec{\alpha}} = \begin{bmatrix} | \\ \vec{r} \\ | \end{bmatrix}$$

we'll use Gold codes for $\vec{s}_i$'s

Gold codes have length 1023 $\quad \vec{s}_i \in \mathbb{R}^{1023}$

$\vec{r} \in \mathbb{R}^{1023}$

max 1023 orthogonal vectors

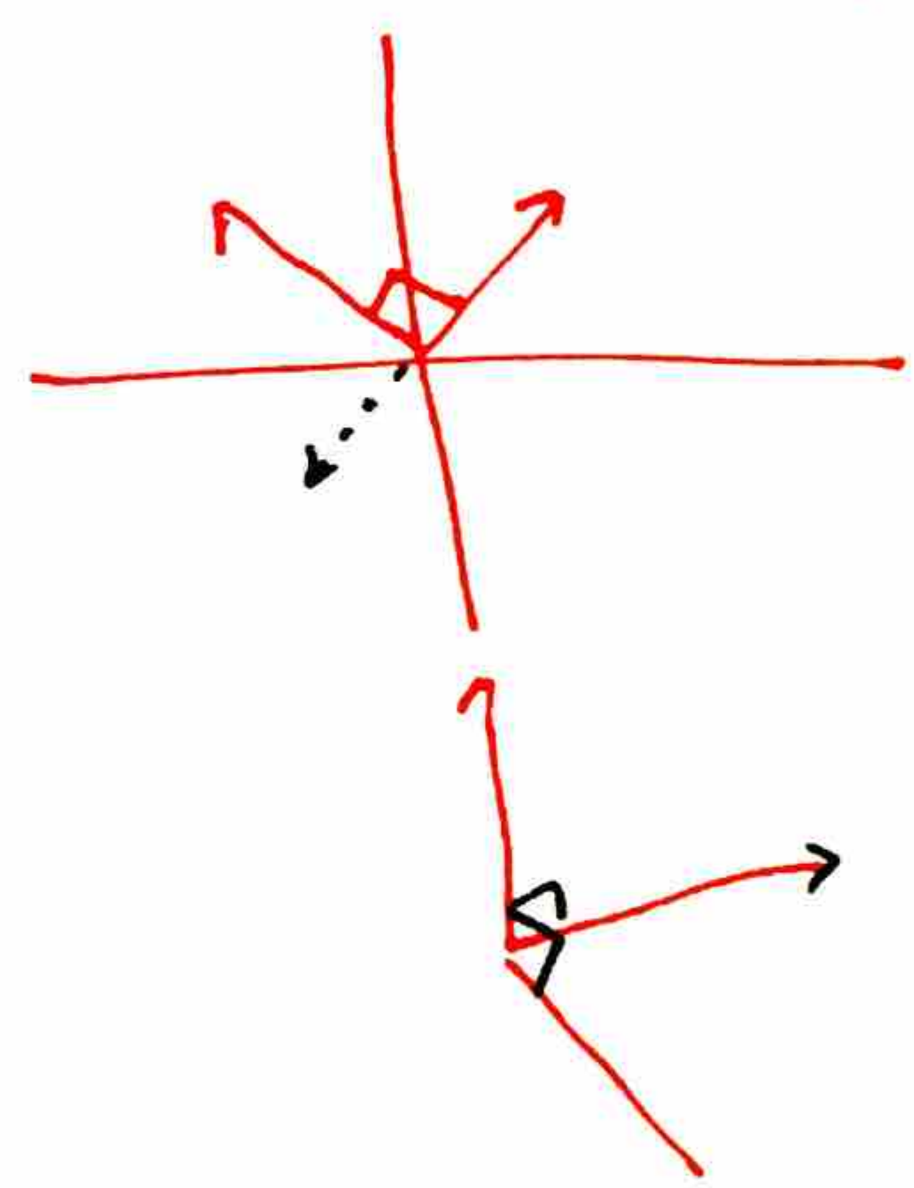
$$S\vec{\alpha} = \vec{r}$$

S: $1023 \times 10{,}000$ $\quad \vec{\alpha}$: $10{,}000$ $\quad \vec{r}$: 1023

* Idea: use addition structure (information) in the problem *

① Gold codes are "nearly orthogonal" with each other

vectors length 2



$\vec{s}_1, \vec{s}_2$ } Gold codes

$\langle \vec{s}_1, \vec{s}_2 \rangle$ is small

$\langle \vec{s}_1, \vec{s}_1 \rangle$ is large $= 1023$

$\vec{s}_1$: [1 -1 -1 ... 1 -1]

$\langle \vec{s}_i, \vec{s}_i \rangle =$ large, 1023

$\langle \vec{s}_i, \vec{s}_{j \neq i} \rangle =$ small

2 most devices are silent most of the time

most $\alpha_i = 0$

$\vec{\alpha}$ is mostly zero

" $\vec{\alpha}$ is a sparse vector "

$\vec{\alpha}$ is k-sparse if there are at most k non-zero elements