EECS 16A
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Lecture 7B
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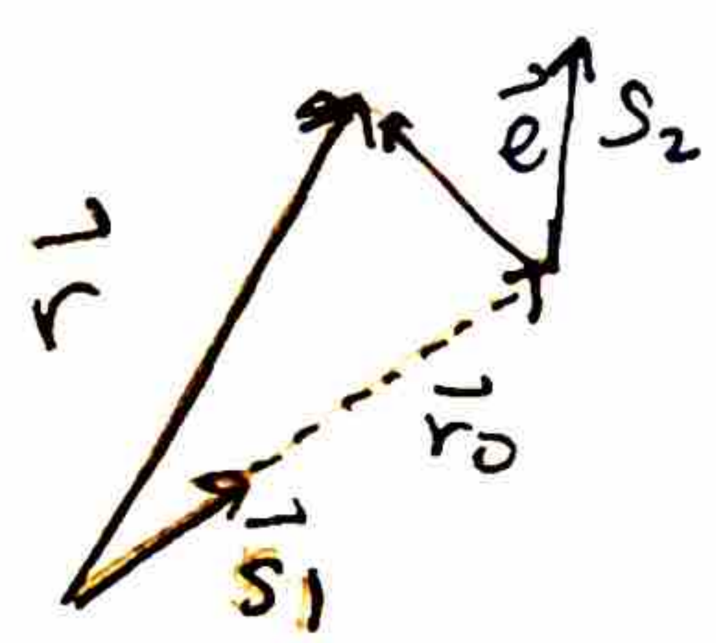
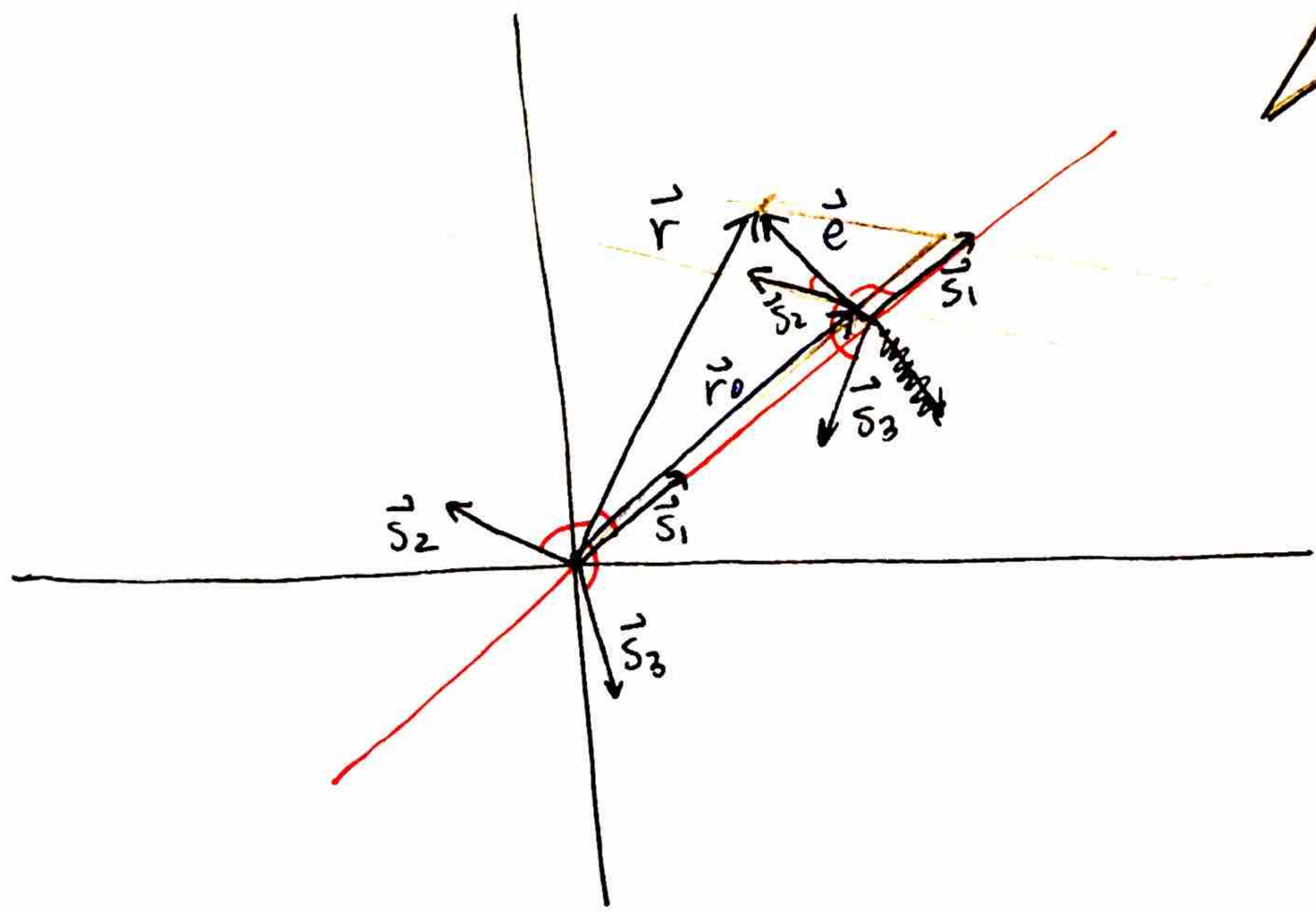
Today:
- more OMP:
  - Graphical interpretation
  - OMP for sparse imaging

course evals: https://course-evaluations.berkeley.edu

Solve $S\vec{\alpha} = \vec{r}$ ← $\vec{r} \in \mathbb{R}^2$

$S = \begin{bmatrix} \vec{s}_1, & \vec{s}_2, & \vec{s}_3 \end{bmatrix}$





① Take inner product of $\vec{r}$ with every $\vec{s}_i$ and find the max abs. of inner product

$\langle \vec{r}, \vec{s}_i \rangle = \underbrace{\|\vec{r}\| \, \|\vec{s}_i\|}_{a} \cos\theta_i = a\cos\theta_i$ $\quad\theta_i$ = angle between $\vec{r}$ and $\vec{s}_i$

→ find θ_i close to zero (or 180°)

This ex. $\vec{s}_1$ has max $\langle \vec{r}, \vec{s}_i \rangle$

② Update matrix ~~A~~ A and vector $\vec{x}$

↳ keep track of which vectors we found

$A = \vec{s}_1$ $\quad \vec{x} = \alpha_1$

No graphical interpretation

③ Project $\vec{r}$ onto $\mathrm{Col}(A)$

→ $\mathrm{span}\{A\} = \mathrm{span}\{\vec{s}_1\}$

④ Update error

$\vec{e} = \vec{r} - \vec{r}_0$

End iter 1.

Start iter 2.

① inner product of $\vec{e}$ with $\vec{s}_i$

↳ expect $\langle \vec{e}, \vec{s}_1 \rangle = 0$

here $\langle \vec{e}, \vec{s}_2 \rangle$ is maximum

② Update A and $\vec{x}$

$A = [\vec{s}_1, \vec{s}_2] \quad \vec{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

③ project $\vec{r}$ onto $\mathrm{col}(A) = \mathrm{span}\{\vec{s}_1, \vec{s}_2\}$

$\hat{x} = (A^T A)^{-1} A^T \vec{r}$

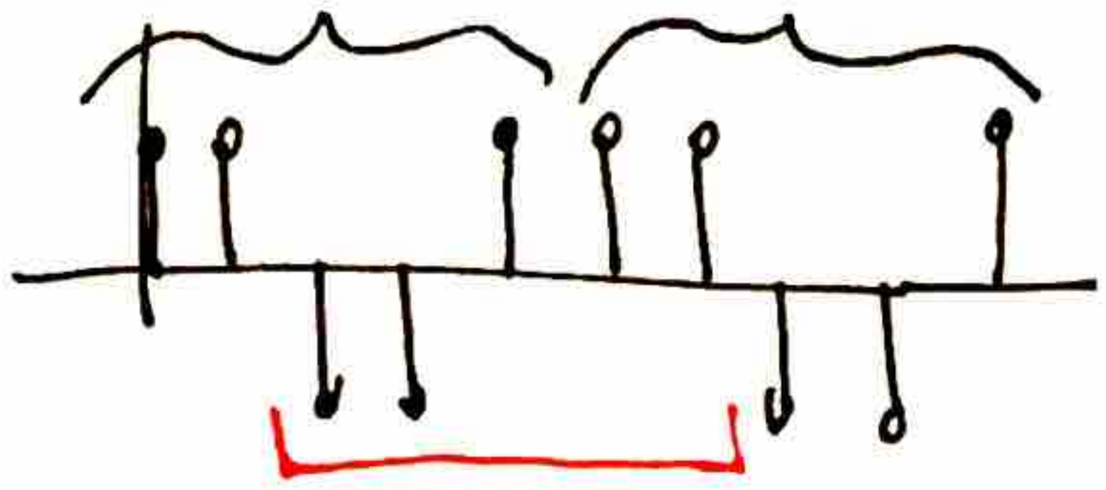
$\vec{r}_0 = A\hat{x}$

④ No more error. We're done!

OMP with Delays

→ more realistic model

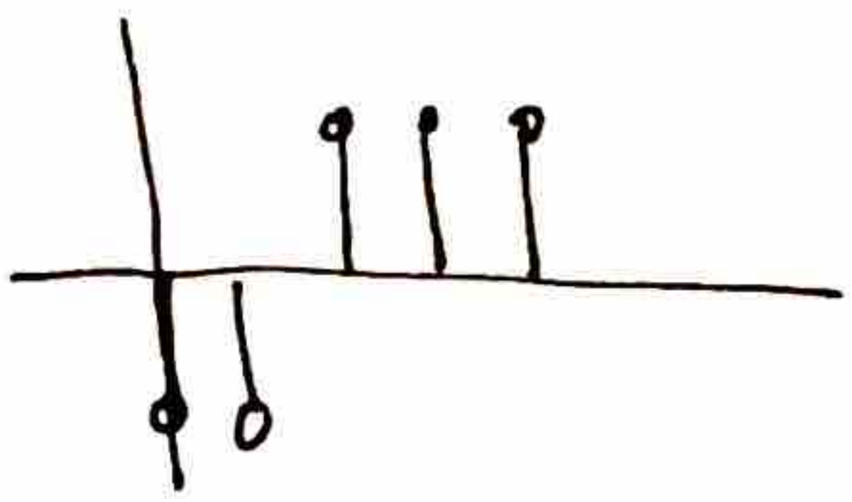
$\vec{s}_1$



→ $\vec{s}_1$ now transmits a periodic signal

→ $\vec{r}$ received signal has some unknown delay

$\vec{r}$ =



$\vec{r} = \alpha_1 \vec{s}_1^{\,(\tau_1)}$

↑ unknown delay

$$\vec{s}_1^{\,(0)} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \qquad \vec{s}_1^{\,(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\vec{s}_1^{\,(2)} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

We can still write this as matrix equation:

$$\vec{r} = \begin{bmatrix} \vec{s}_1^{\,(0)} & \vec{s}_1^{\,(1)} & \dots & \vec{s}_1^{\,(4)} \end{bmatrix} \begin{bmatrix} \\ \vec{\alpha}' \\ \\ \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ \alpha_1 \\ 0 \\ 0 \end{bmatrix}$$

→ OMP: take inner product w/ $\vec{r}$ and every column
↳ now columns are shifted versions
↳ same as correlation.

What if we have 2 devices

$$\vec{r} = \alpha_1 \vec{s}_1^{(\tau_1)} + \alpha_2 \vec{s}_2^{(\tau_2)}$$

$$\vec{r} = \begin{bmatrix} \vec{s}_1^{(0)} & \vec{s}_1^{(1)} & \cdots & \vec{s}_1^{(4)} & \vec{s}_2^{(0)} & \vec{s}_2^{(1)} & \cdots & \vec{s}_2^{(4)} \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ \alpha_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_2 \\ 0 \\ 0 \end{bmatrix}}_{\vec{\alpha}'} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\\0\\0\end{matrix}}\right\} \text{5 elements} \\ \left.\vphantom{\begin{matrix}0\\0\\0\\0\\0\end{matrix}}\right\} \text{5 elements} \end{matrix}$$

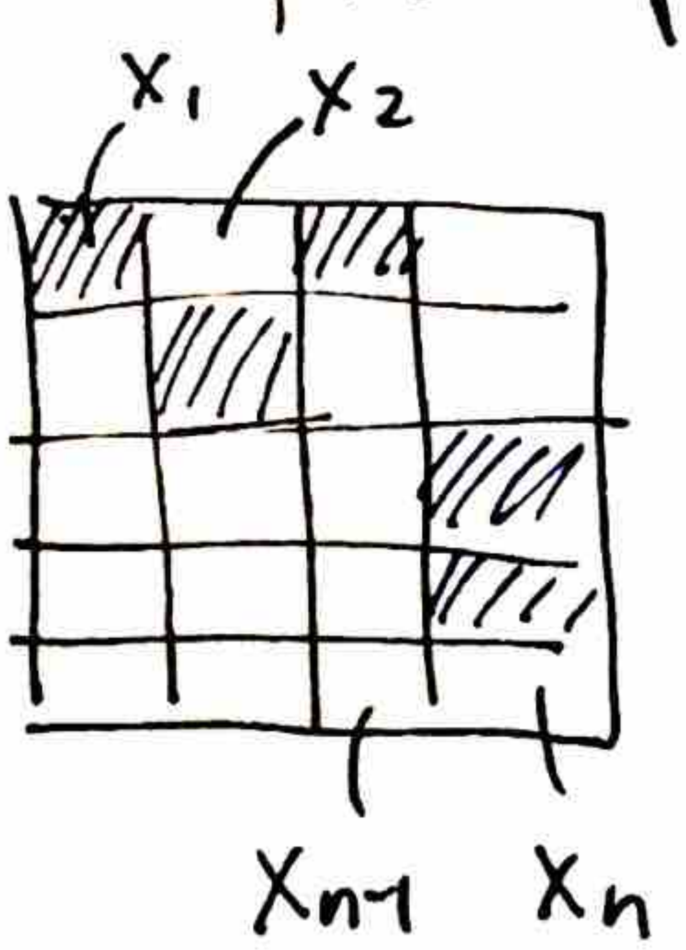
Same as correletion between
$\vec{r}$ and $\vec{s}_1$
$\vec{r}$ and $\vec{s}_2$

$$\vec{r} = \alpha_{11} \vec{s}_1^{(0)} + \alpha_{12} \vec{s}_1^{(1)} + \alpha_{13} \vec{s}_1^{(2)} + \ldots + \alpha_{15} \vec{s}_1^{(4)} + \alpha_{21} \vec{s}_2^{(0)} + \alpha_{22} \vec{s}_2^{(1)} + \ldots + \alpha_{25} \vec{s}_2^{(4)}$$

$\vec{s}_1^{(4)}$

$\vec{s}_2^{(4)}$

Example of OMP : Sparse Imaging



$$\underbrace{\begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ \\ \end{bmatrix}}_{\substack{\vec{m} \\ \text{measurements}}} = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 0 \\ & & A & \\ & & & \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\text{image}}$$

How many measurements for n pixels?
→ Module 1 : n measurements

But, what if we know that the image is mostly zero? (image is sparse?)

- $\vec{x}$ is sparse
- we can design A so that columns are nearly orthogonal

ex) night sky
fluorescent image

Example:

x_1	x_2	x_3
x_4	x_5	x_6

6 pixel image

We know that 2 pixels are non-zero
We're going to take 4 measurements.

$$\begin{bmatrix}3\\2\\0\\5\end{bmatrix} = \vec{m} = \begin{array}{c}\begin{matrix}\vec{a}_1 & \vec{a}_2 & & & & \vec{a}_6\end{matrix}\\ \underbrace{\begin{bmatrix}1 & 0 & 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 1 & 1 & 0\\ 0 & 1 & 0 & 1 & 0 & 1\\ 0 & 1 & 1 & 0 & 1 & 0\end{bmatrix}}_{A}\end{array} \vec{x}$$

ITER 1

① $\langle \vec{m}, \vec{a}_i \rangle$

i	$\langle \vec{m}, \vec{a}_i \rangle$
1	5
2	~~8~~ 5
3	8
4	2
5	7
6	3

$i = 3$ maximizes the inner product

② Update a matrix w/ our column $B = \vec{a}_3$

3 Project $\vec{m}$ onto the columnspace (B)

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{m} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 5 \end{bmatrix}$$

$$\vec{m}_0 = B(B^T B)^{-1} B^T \vec{m}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} (2)^{-1} (8)$$

$$= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix}$$

4 Update error

$$\vec{e} = \vec{m} - \vec{m}_0 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

ITER 2

1 calculate $\langle \vec{a}_i, \vec{e} \rangle$

i	$\langle \vec{e}, \vec{a}_i \rangle$
1	1
2	1
3	0
4	2
5	3
6	-1

$\leftarrow$ $i = 5$ maximizes $\langle \vec{e}, \vec{a}_i \rangle$

2 Update B

$$B = [\vec{a}_3, \vec{a}_5] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

3 Project $\vec{m}$ on to col (B)

$$\vec{m}_0 = B \begin{bmatrix} \hat{x}_3 \\ \hat{x}_5 \end{bmatrix} \qquad \begin{bmatrix} \hat{x}_3 \\ \hat{x}_5 \end{bmatrix} = (B^T B)^{-1} B^T \vec{m}$$

$$\begin{bmatrix} \hat{x}_3 \\ \hat{x}_5 \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \\ 5 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 8 \\ 7 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{m}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 5 \end{bmatrix} = \vec{m}$$

$$\vec{e} = \vec{m} - \vec{m}_0 = \vec{0}$$