EECS 16A
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Review session
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Review: Proofs

Tips for doing proofs

- don't be scared
- writing things down!
  - → write precise definitions
  - → rewrite things in words in math notation
  - → simple example
    - → n variables → try 2 variables
    - → variables → plug in some numbers
  - → work from start + end

    what we know:

    look for connections between them

    what we want to show:
  - → write down definitions ~~ab~~ or facts that might be related
- You should understand all the steps in your proof
  - → your reader should understand
  - → add some justification

Types of proofs: (that we've introduced)

- Direct proof: series of mathematical steps.
  ↑ most common in this class

- Constructive proofs:
  "Show that there exists..."
  - give an example of something that meets the requiirements

- Proof by contradiction:
  " Show there does not exist ...."
  ___☆___ does not exist

  → assuming ☆ does exist
  → use this assumption
  → find a contradiction
  ex) $10 = 0$
  ex) contradiction w/ a different assumption.
  $\{\vec{v}_1 \dots \vec{v}_n\}$ are linearly dependent
  $\{\vec{v}_1 \dots \vec{v}_n\}$ are linearly indendent

Example 1

Let $\vec{x}$ be orthogonal to $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$.
Prove that $\vec{x}$ is orthogonal to any vector in the span $\{\vec{a}_1, \vec{a}_2, \dots \vec{a}_n\}$

$\boxed{\text{Proof}}$

By the def. of span

$\vec{v} \in \text{span}\{\vec{a}_1 \dots \vec{a}_n\}$ we know

if $\vec{v} = c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n$ } representative vector in the span $\{\vec{a}'s\}$

Def of orthogonaity } want to show

$\langle \vec{x}, \vec{v} \rangle = 0$

$\vec{x}^T\vec{v} = 0$

we also know

$$\left.\begin{array}{ll} \langle \vec{x}, \vec{a}_1 \rangle = 0 & \vec{x}^T\vec{a}_1 = 0 \\ \vdots & \vdots \\ \langle \vec{x}, \vec{a}_n \rangle = 0 & \vec{x}^T\vec{a}_n = 0 \end{array}\right\} \text{we know}$$

Try:

$$\begin{aligned} \langle \vec{x}, \vec{v} \rangle &= \langle \vec{x}, c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n \rangle \\ &= \langle \vec{x}, c_1\vec{a}_1 \rangle + \langle \vec{x}, c_2\vec{a}_2 \rangle + \dots + \langle \vec{x}, c_n\vec{a}_n \rangle \\ &= c_1 \underbrace{\langle \vec{x}, \vec{a}_1 \rangle}_{=0} + c_2 \underbrace{\langle \vec{x}, \vec{a}_2 \rangle}_{=0} + \dots + c_n \underbrace{\langle \vec{x}, \vec{a}_n \rangle}_{=0} \\ &= 0 \end{aligned}$$

We've shown $\vec{x}$ is orthogonal to $\vec{v}$. Since $\vec{v}$ represents every vector in span $\{\vec{a}_1 \dots \vec{a}_n\}$ we've show $\vec{x}$ is orthogonal to span $\{\vec{a}_1 \dots \vec{a}_n\}$.

Example 2 [MT 1 Fall 2019]

$\boxed{\text{THM}}$ If A has a non-trival nullspace, then A is not invertible.

$\boxed{\text{Proof}}$

Given: $A\vec{v} = \vec{0}$ $\quad \vec{v} \neq \vec{0}$ } know

Want to show: A^{-1} does not exist
↳ hard.

Assume A^{-1} exists. } "know"

Try: $A^{-1}A\vec{v} = A^{-1}\vec{0}$ ← left mult. by A^{-1}

$\vec{v} = \vec{0}$ ← doing math / def. of inverse

Give a contradiction.

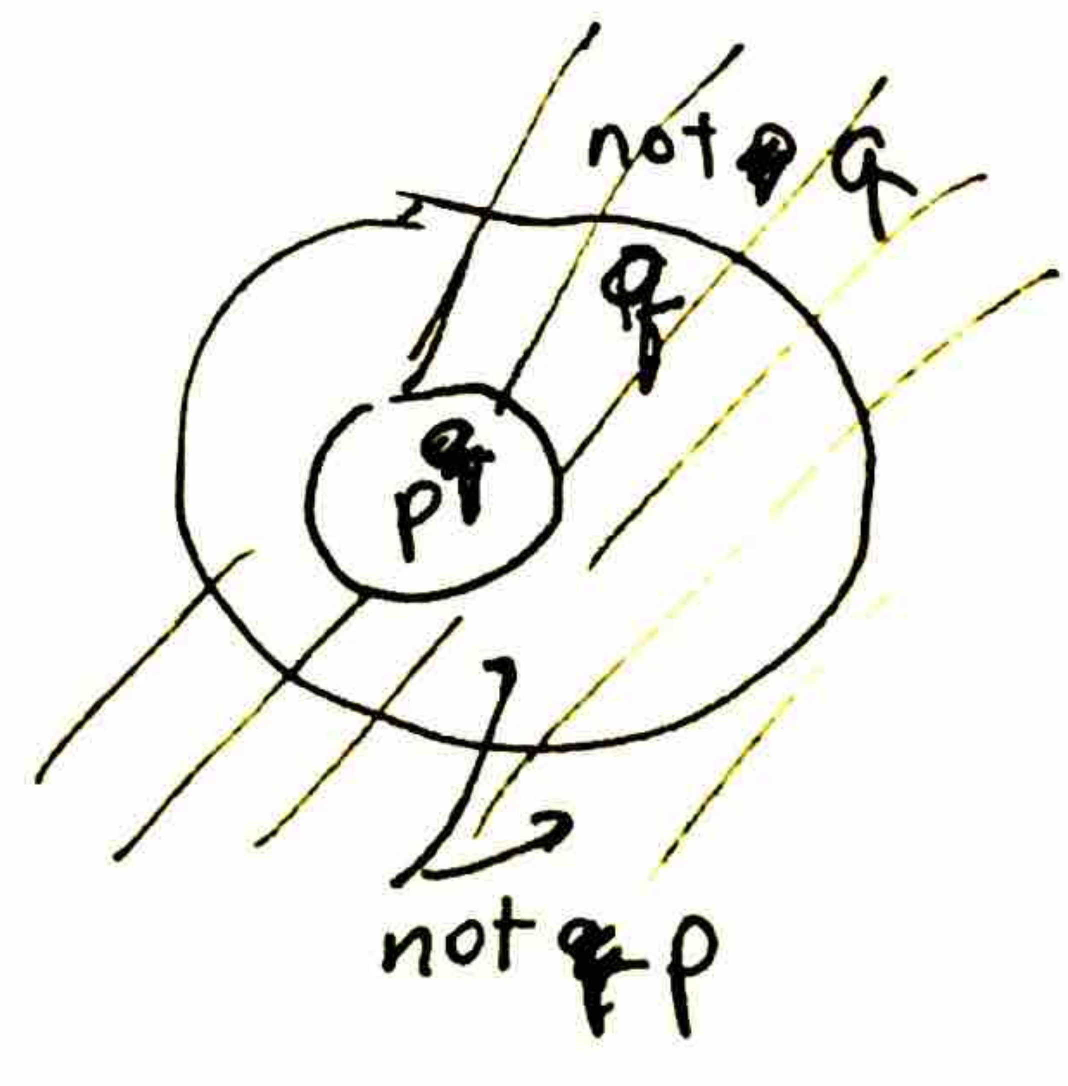
Therefore A^{-1} must not exist.

Contrapositive

if p then q equivalent to

if not q then not p

if A is invertible then A has a trivial nullspace.



Example 3

If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $\boxed{\vec{u} + \vec{v} + \vec{w} = \vec{0}}$

Prove $\text{span}\{\vec{u}, \vec{v}\} = \text{span}\{\vec{v}, \vec{w}\}$.

(1) if $\vec{x} \in \text{span}\{\vec{u}, \vec{v}\}$ then $\vec{x} \in \text{span}\{\vec{v}, \vec{w}\}$

(2) if $\vec{y} \in \text{span}\{\vec{v}, \vec{w}\}$ then $\vec{y} \in \text{span}\{\vec{u}, \vec{v}\}$

Aside:
"if and only if"
→ prove both directions

(1) $\vec{x} = \alpha_1 \vec{u} + \alpha_2 \vec{v}$ for some α_1, α_2 Def of span

$\vec{u} = -\vec{v} - \vec{w}$

$\vec{x} = \alpha_1(-\vec{v} - \vec{w}) + \alpha_2 \vec{v}$

$\vec{x} = -\alpha_1 \vec{w} - \alpha_1 \vec{v} + \alpha_2 \vec{v}$

$\vec{x} = (\alpha_2 - \alpha_1)\vec{v} - \alpha_1 \vec{w}$ lets set $\beta_1 = \alpha_2 - \alpha_1$, $\beta_2 = -\alpha_1$

want to show: $\vec{x} = \beta_1 \vec{u} + \beta_2 \vec{w}$ for some β_1, β_2

(2) $\vec{y} = \beta_1 \vec{v} + \beta_2 \vec{w}$ for some β_1, β_2

$\vec{w} = -\vec{u} - \vec{v}$

$\vec{y} = \beta_1 \vec{v} + \beta_2(-\vec{u} - \vec{v})$

$\vec{y} = \beta_1 \vec{v} - \beta_2 \vec{u} - \beta_2 \vec{v}$

$\vec{y} = -\beta_2 \vec{u} + (\beta_1 - \beta_2)\vec{v}$ let $\alpha_1 = -\beta_2$, $\alpha_2 = \beta_1 - \beta_2$

want to show: $\vec{y} = \alpha_1 \vec{u} + \alpha_2 \vec{v}$ for some α_1, α_2

Example 4

Let $P, Q \in \mathbb{R}^{n\times n}$ } square

If Q has rank n and $PQ = O$ (where O is a matrix ($n\times n$) of all zeros), prove that P must be all zeros.

Proof

Since Q is full rank, then we know Q is invertible.

$$PQQ^{-1} = OQ^{-1}$$ ~~left~~ right mult. by Q^{-1}

$$P = O$$

Let columns of Q be $\vec{q}_1 \dots \vec{q}_n$

we know by def. of mat. multiplication

$P\vec{q}_i = \vec{0}$ $\qquad \alpha_1 P\vec{q}_1 + \alpha_2 P\vec{q}_2 + \dots \alpha_n P\vec{q}_n = \vec{0}$

$$P(\alpha_1\vec{q}_1 + \dots + \alpha_n \vec{q}_n) = \vec{0}$$

since $\text{span}\{\vec{q}_1 \dots \vec{q}_n\} = \mathbb{R}^n$

$$P\vec{x} = \vec{0} \text{ for any } \vec{x}$$

Assume $P \neq O$. Then there is at least one column $\vec{p}_i \neq \vec{0}$. If $\vec{x} = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}$ ← ith position

$P\vec{x} = \vec{p}_i$ contradiction with $P\vec{x} = \vec{0}$ for all $\vec{x}$

so $P = O$.