Exam location: 145 Dwinelle, last SID# 2

PRINT your student ID: ____

PRINT AND SIGN your name: __________________, __________________ ______________________
(last) (first) (signature)

PRINT your Unix account login: ee16a-_____________

PRINT your discussion section and GSI (the one you attend): ____

Name and SID of the person to your left: ____

Name and SID of the person to your right: ____

Name and SID of the person in front of you: ____

Name and SID of the person behind you: ____

Section 0: Pre-exam questions (3 points)

1. **What other courses are you taking this term? (1 pt)**

2. **What activity do you really enjoy? Describe how it makes you feel. (2 pts)**

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

Section 1 (18 points)

Unless told otherwise, you must show work to get credit. There will be very little partial credit given in this section.

3. True/False (6 points, 1 point for each question)

Answer each of the following questions by circling True or False. No work is necessary for credit on this part.

(a) **(True) (False)** The pivot columns of matrix A forms a basis for the column space of A. **Solutions:** True

(b) **(True) (False)** Let A be a 2×2 matrix, where $A^2 = 0$. Then A is the zero matrix. **Solutions:** False

(c) **(True) (False)** Let A, B, C be some arbitrary matrices. Then, $(AB)C = A(BC)$. **Solutions:** True

(d) **(True) (False)** An $M \times N$ matrix has at most N pivots. **Solutions:** True

(e) **(True) (False)** $AB = BA$ where A and B are $N \times N$ matrices. **Solutions:** False

(f) **(True) (False)** Applying any pair of 2×2 rotation matrices to an input vector is a commutative operation. **Solutions:** True

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4. Proof (7 points)

(a) Prove that if $A\vec{x} = 0$ for some nonzero $\vec{x}$, then the columns of A are linearly dependent.

Solutions: $A\vec{x}$ is a linear combination of the columns of A (call these $\vec{a}_i$).
Pick the nonzero $\vec{x}$ such that $A\vec{x} = 0$, then: $\sum_i x_i \vec{a}_i = 0$
Which is proof of the linear dependence of $\vec{a}_i$.

(b) Prove that if $A^2 = 0$ where A is an arbitrary square matrix, then the columns of A are linearly dependent.

Solutions: Using the same notation from the previous part, A^2 is A applied on $\vec{a}_i$.
If any $\vec{a}_i$ are $\vec{0}$, we know immediately that the columns are linearly dependent.
Otherwise, we have that $A\vec{a}_i$=0, which is proof of linear dependence of the columns from part a.

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5. Inverse of a Matrix (5 points)

Find the inverse of A, if it exists. If not, explain why. $A = \begin{bmatrix} 5 & 4 & 2 \\ 1 & 2 & 1 \\ 9 & 6 & 3 \end{bmatrix}$.

Solutions:

This matrix is **singular/noninvertible**, and this can be found in a number of ways.

One is that the third row is a linear combination of the two other rows - precisely, $R_3 = 2R_1 - R_2$. Therefore, by the Invertible Matrix Theorem, this matrix is noninvertible.

Alternatively, this could be found by row reducing this matrix alongside an identity matrix, and stopping when the third row is reduced to entirely zeroes. A 3×3 matrix with a row of zeroes is at most of rank 2, and thus not invertible.

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Section 2 (55 points)

6. Directional Shovels (10 points)

Kody and Nara were found exceptional at taking measurements to figure out light intensities, and they were both granted admission to a graduate school. Unfortunately, they both supported their new school's football team while they were playing against Berkeley and angry Berkeley fans found them and left them in a room at an unknown location under the ground. As compassionate people, Berkeley fans left some tools in the room that can help them escape.

(a) Kody found a shovel in the room and figured that it can operate in the following directions: $\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\}$. Is it possible for them to escape to Berkeley by digging in the given directions to a point which is located at $\begin{bmatrix}3\\-2\\5\end{bmatrix}$ given that they are at point $\begin{bmatrix}0\\0\\0\end{bmatrix}$? If so, find the scalars that multiply the vectors such that they reach Berkeley.

Solutions: (3,-2). $3\begin{bmatrix}1\\0\\1\end{bmatrix} - 2\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}3\\-2\\5\end{bmatrix}$

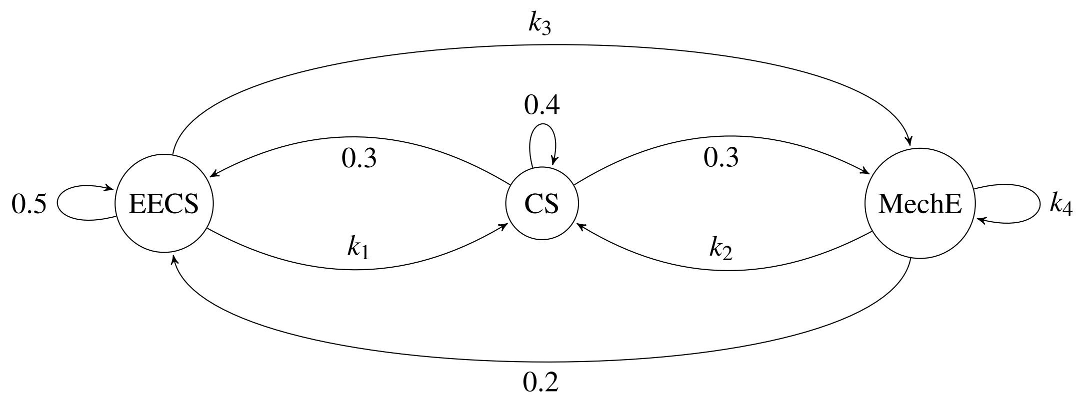
(b) While Kody was busy planning his escape to Berkeley, Nara found a pick-axe in the room that can operate in the following directions: $\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \}$. Nara is convinced that the axe she found is better, but Kody disagrees. Show that Kody's shovel can reach anywhere that Nara's pick-axe can.

Solutions: Put these 3 vectors as row vectors in a matrix and row reduce. Notice that it reduces to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ Similar to the proof in the last question of discussion 3A. This shows that row spaces are equivalent.

Another solution is show that all of Nara's directions can be reached by Kody's vectors. It is easy enough to find the correct linear combinations by inspection: $(2,2),(1,-1),(3,-2)$

7. Graph Majors (30 points)

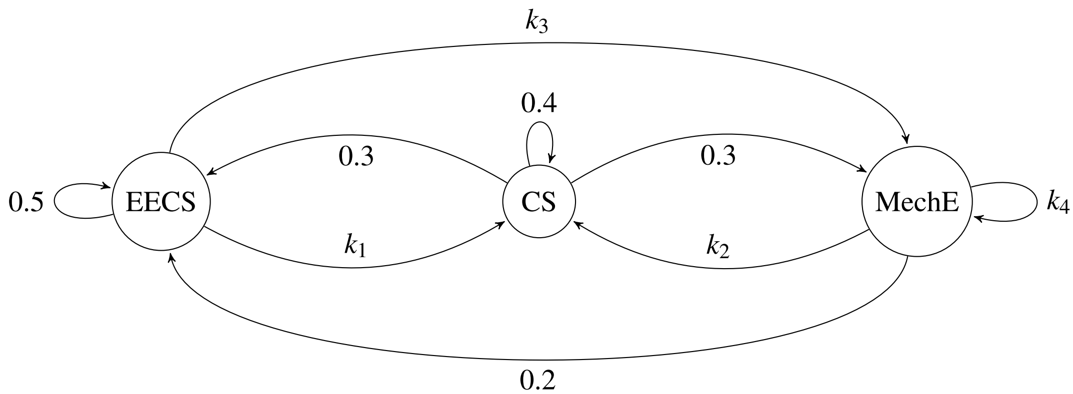
We'd like to understand how engineering undergrads change their majors. For simplicity, there are three majors we'll look at: EECS, CS, and MechE. Let's assume that students can only be studying one major at a time, and must be studying one of these three majors. Let's also assume that once a week, students can choose to switch to another major, or stick with what they're studying. So, a discrete time step represents one week.



At the start of week n, the number of EECS, CS, MechE students are $x_e[n]$, $x_c[n]$, and $x_m[n]$, respectively. Let $\vec{x}[n] = \begin{bmatrix} x_e[n] \\ x_c[n] \\ x_m[n] \end{bmatrix}$. Also let $\vec{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}$.

(a) Write the transition matrix, A, such that $\vec{x}[n+1] = A\vec{x}[n]$.

Solutions: $A = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ k_1 & 0.4 & k_2 \\ k_3 & 0.3 & k_4 \end{bmatrix}$



(b) Assume that from one week to the next, no students drop out or are enrolled to the system – in other words, the total number of students is conserved. Write a system of four linear equations that relate k_1, k_2, k_3, k_4. *Hint: you should use* $x_e[n]$, $x_c[n]$, $x_m[n]$, $x_e[n+1]$, $x_c[n+1]$, $x_m[n+1]$ *in your answer.*

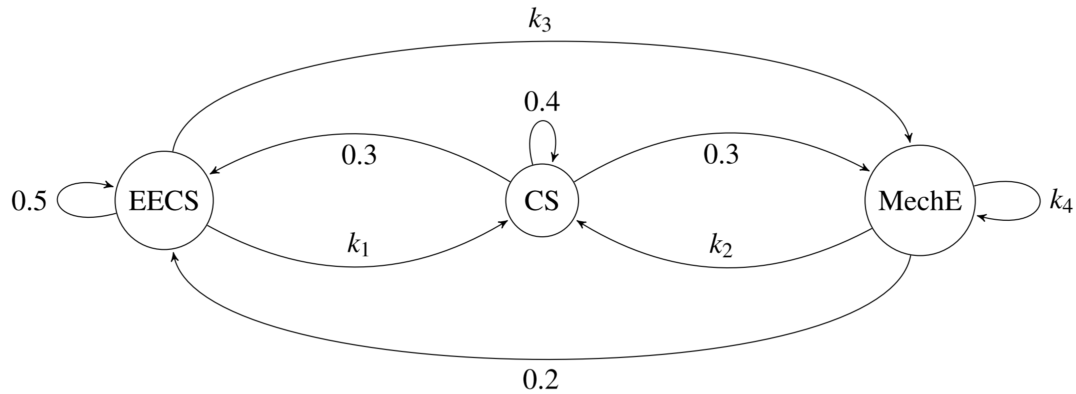
Solutions:

By $\vec{x}[n+1] = A\vec{x}[n]$, we derive the equations:

$k_1 x_e[n] + 0.4 x_c[n] + k_2 x_m[n] = x_c[n+1]$
$k_3 x_e[n] + 0.3 x_c[n] + k_4 x_m[n] = x_m[n+1]$

By conservation of students, we derive the equations:

$0.5 + k_1 + k_3 = 1$
$0.2 + k_2 + k_4 = 1$



(c) Let $\vec{x}[10] = \begin{bmatrix} 100 \\ 200 \\ 200 \end{bmatrix}$ and $\vec{x}[11] = \begin{bmatrix} 150 \\ 100 \\ 250 \end{bmatrix}$. Rewrite your four linear equations from part (b) in the form $T\vec{k} = \vec{b}$, where $\vec{k}$ is the vector defined above and $\vec{b}$ is a vector of constants. Do **not** solve for $\vec{k}$.

Solutions:

Plugging in numerical values to the above solution:

$100k_1 + (0.4)(200) + 200k_2 = 100$

$100k_3 + (0.3)(200) + 200k_4 = 250$

$0.5 + k_1 + k_3 = 1$

$0.2 + k_2 + k_4 = 1$

In the form $T\vec{k} = \vec{b}$, we have

$$\begin{bmatrix} 100 & 200 & 0 & 0 \\ 0 & 0 & 100 & 200 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 20 \\ 190 \\ 0.5 \\ 0.8 \end{bmatrix}$$

(d) Now let us redefine our graph transition matrix A such that $A = \begin{bmatrix} 0.6 & 0.4 & 0.2 \\ 0.3 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$. Given $\vec{x}[923]$, is it possible to find $\vec{x}[2]$? Give a mathematical justification and a brief explanation of how. Do not make any assumptions derived from previous parts of this problem.

Solutions:

Yes, because given those values for A, A is invertible.

This can be shown by row reducing to see that there is a pivot in every row and column of A:

Multiplying each row of A by ten gives:

$$\begin{bmatrix} 6 & 4 & 2 \\ 3 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$$

Swapping R_1 and R_3 gives:

$$\begin{bmatrix} 1 & 4 & 5 \\ 6 & 4 & 2 \\ 3 & 2 & 3 \end{bmatrix}$$

$R_2 - 2 * R_3$ gives:

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 0 & -4 \\ 3 & 2 & 3 \end{bmatrix}$$

$R_3 - 3 * R_1$ gives:

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 0 & -4 \\ 0 & -10 & -12 \end{bmatrix}$$

From here, we see there is a nonzero pivot in every row and column in A.

Another way to show that A is invertible would be to calculate A^{-1}. This can be done by row reducing $A|I$ (this calculation is more involved). A^{-1} is found to be:

$$A^{-1} = \begin{bmatrix} 0.5 & 3 & -2\frac{1}{3} \\ 3 & -7 & 3 \\ -2.5 & 5 & 0\frac{2}{3} \end{bmatrix}$$

This means you can back infer based on the equation $A^{-1}\vec{x}[n+1] = \vec{x}[n]$.

(e) Let us redefine A as $A = \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.4 & 0.7 & 0.1 \\ 0.4 & 0.2 & 0.6 \end{bmatrix}$. Is $\vec{x}[5] = \begin{bmatrix} 120 \\ 120 \\ 260 \end{bmatrix}$ a valid state for this system? Explain. Assume the states begin with some $\vec{x}[0]$, where $\vec{x}[0]$ is not the zero vector $\vec{0}$. Do not make any assumptions derived from previous parts of the problem.

Solutions:

No, because the third row is two times the first row of A. Thus, at any state that is not the initial state, $2 * x_e[t] = x_m[t]$ must be true by the state transitions. The given $\vec{x}[5]$ violates this.

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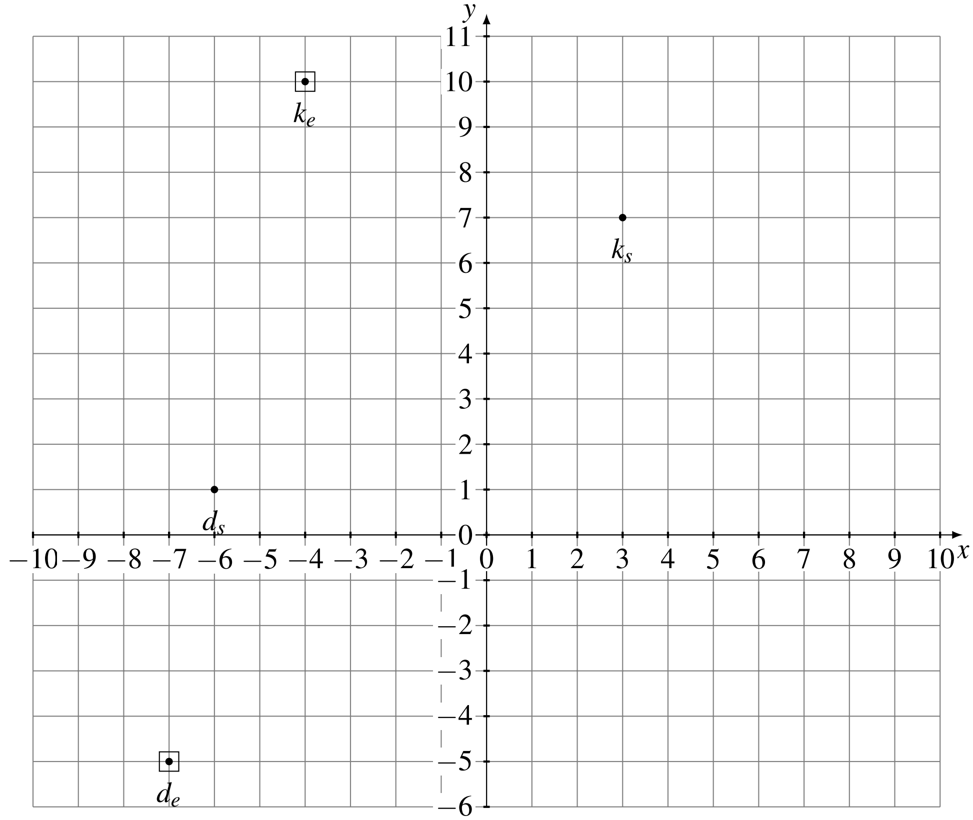
8. Transformation Basketball (15 points)

Kevin Bancroft just joined the Column Space Warriors. In order to better learn how to cooperate with the team before the season starts, he and his teammates are practicing some basketball drills.

(a) Kevin Bancroft and Draymond Blue-Gold are running a drill where they each have to run from a starting coordinate to an end coordinate. Kevin starts at point $k_s = [3\ 7]^T$ and wants to go to point $k_e = [-4\ 10]^T$. Draymond starts at point $d_s = [-6\ 1]^T$ and wants to go to point $d_e = [-7\ -5]^T$.



Each player must apply the same matrix transformation A on his starting point to reach reach his end point, such that $Ak_s = k_e$ and $Ad_s = d_e$. Derive the transformation matrix A, if possible. We also know that the transformation matrix A is of the form $A = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$, where a, b are real numbers.

Solutions:

$$\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

So we can plug in our known values and solve the system.

$$3 + 7a = k_e x = -4 \implies a = -1$$

$$3b + 7 = k_e y = 10 \implies b = 1$$

$$T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

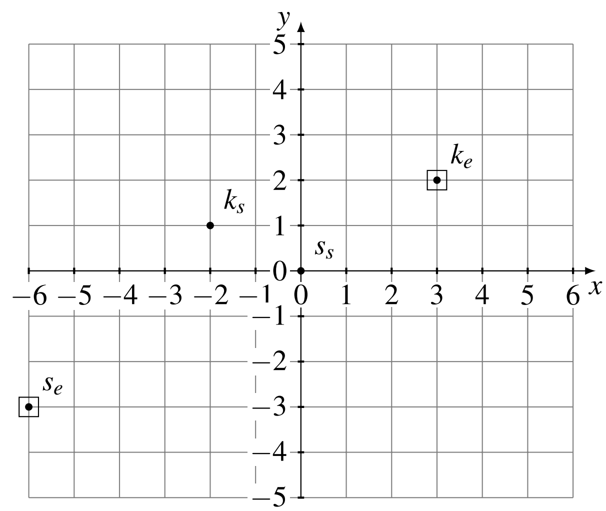
(b) Steph Current noticed Kevin and Draymond running this drill, and decided to join them. For their next move, they will be using transformation matrix B.

$$B = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$$

Describe what transformation matrix B performs to an input position in terms of rotations, scaling, and reflections.

Solutions: After applying the matrix to some sample coordinates such as the standard basis vectors, we can see that the matrix rotates by -45 and scales by $2\sqrt{2}$

(c) After a couple of drills, Kevin Bancroft came up with a new idea – he decided to race his teammates across the court to see who is faster. Kevin starts at point $k_s = [-2\ 1]^T$ and ends at $k_e = [3\ 2]^T$. Steph starts at $s_s = [0\ 0]^T$ and ends at $s_e = [-6\ -3]^T$. Can this be represented by a transformation matrix? Briefly justify why or why not.



Solutions: It cannot because in the case of Steph [0 0] goes to a nonzero vector. This transformation cannot be a linear transformation, so there is no matrix representation.