EECS 16A   Designing Information Devices and Systems I
Fall 2019
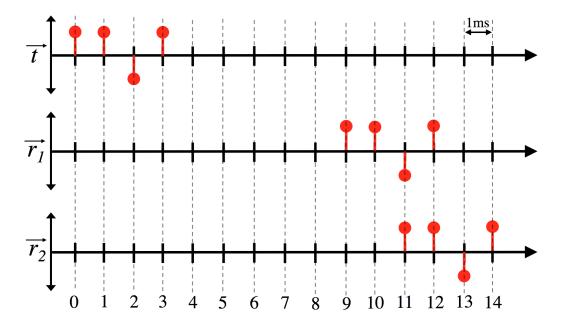# Final Exam

<div style="text-align:center">

Final Solution

</div>

PRINT your student ID: _____

PRINT AND SIGN your name: _____, _____ _____
<div style="text-align:center">(last name)   (first name)   (signature)</div>

PRINT your discussion section and GSI(s) (the one you attend): _____

Name and SID of the person to your left: _____

Name and SID of the person to your right: _____

Name and SID of the person in front of you: _____

Name and SID of the person behind you: _____

1. **Tell us about something you are proud of this semester. (2 Points)**

2. **What are you looking forward to over winter break? (2 Points)**

Do not turn this page until the proctor tells you to do so. You may work on the questions above.
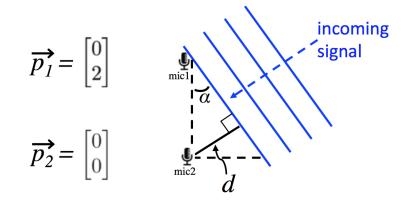
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3. **Where is the sound coming from?** **(14 points)** (All subparts of this problem can be solved independently.)

In this problem we will use concepts from the class to determine the angle and position of an incoming audio signal recorded by microphones. All of the microphones and transmitting sources are in the same plane (i.e. the problem is in 2D).

(a) (4 points) A transmitter sends the signal $\vec{t}$. This signal is received by microphone 1 as $\vec{r}_1$ and by microphone 2 as $\vec{r}_2$. The microphones record 1 sample every millisecond ($1ms = 10^{-3}$ s) and the speed of sound is $v_s = 300m/s$. **How far are microphone 1 and microphone 2 from the transmitter?**

Hint: You do not have to do cross-correlation to solve this question.

Solution: The source of signal $\vec{t}$ is 9ms $\times$ 300m/s $= 2.7$ meters from microphone 1 and 11ms $\times$ 300m/s $= 3.3$ meters from microphone 2.

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(b) (3 points) Microphone 1 and Microphone 2 receive signals $\vec{r}_1$ and $\vec{r}_2$ respectively from the transmitter. The time delay between $\vec{r}_1$ and $\vec{r}_2$ is related to the perpendicular distance d between Microphone 2 and the incoming signal $\vec{t}$. **You have measured that** $d = 1m$ (i.e. the distance between Microphone 2 and the incoming signal $\vec{t}$) in the figure below. The positions $\vec{p}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\vec{p}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ of Microphones 1 and 2, respectively, are shown below. The units are in meters.

What is the angle of arrival α (see the figure below) between the incoming signal $\vec{t}$ and the line joining the microphones? You may leave your answer in terms of a trigonometric function.

Solution: $\alpha = \arcsin(\frac{d}{2-0}) = \arcsin(\frac{1}{2}) = \frac{\pi}{6} = 30°$

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(c) (7 points) **Now you are considering a different setup of the microphones from earlier parts.** You have placed four microphones at $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$, and $\vec{p}_4$ in the plane (see figure). You want to determine the position of the transmitting source. You used cross correlation and determined that the distances d_i between the microphones and the transmitting source as follows:

$$\vec{p_1} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \quad d_1 = 1\text{m}$$

$$\vec{p_2} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad d_2 = \sqrt{5}\text{m}$$

$$\vec{p_3} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad d_3 = \sqrt{10}\text{m}$$

$$\vec{p_4} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad d_4 = \sqrt{17}\text{m}$$
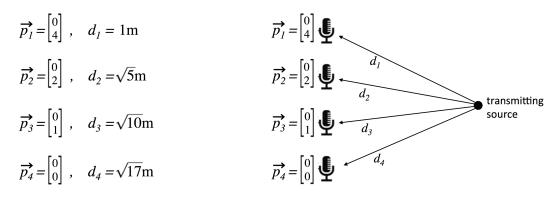

Figure 3.1: Note the microphone schematic is not drawn to scale.

Set up system of linear equations to compute the location of the transmitting source. If you can solve the system to identify the location of the source, solve it. If you cannot identify the location of the source, explain why. Then, propose a design/setup that would make the problem solvable, and explain why your design works.

Solution:

Just as in lecture, we can use the distances measured to write out four equations

$$x^2 + (y-4)^2 = 1 \tag{1}$$
$$x^2 + (y-2)^2 = 5 \tag{2}$$
$$x^2 + (y-1)^2 = 10 \tag{3}$$
$$x^2 + y^2 = 17 \tag{4}$$

Subtract Eq. 4 from all of the other equations to remove the quadratic terms. This gives:

$$-8y + 16 = 1 - 17 = -16 \tag{5}$$
$$-4y + 4 = 5 - 17 = -12 \tag{6}$$
$$-2y + 1 = 10 - 17 = -7 \tag{7}$$

We can already see from the equations above that there are no constraints on the x coordinate in the equations above.

We can write this as the following 2×2 system.

$$\begin{bmatrix} 0 & -8 \\ 0 & -4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -32 \\ -16 \\ -8 \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ? \\ 4 \end{bmatrix} \tag{9}$$

Since the first column here is a column of zeros, the columns of the matrix are linearly dependent, and this least-squares problem does not have a solution. You can recover that $y = 4$, however, substituting back in the equations this does not give you a unique value of x. Since x only appears as a quadratic term $x = +1$ and $x = -1$ are both solutions.

To prevent this from happening you need to make sure your design/microphone setup is such that the columns of the matrix are not linearly dependent. To do this, you just have to place the microphones such that they are not co-linear. e.g. place $\vec{p}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{p}_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ would work, but there are many other correct solutions.

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4. Building a classifier (22 points)

We used least squares to classify data in the "Labeling Patients Using Gene Expression Data" homework problem. We would like to now develop a classifier to classify points based on their distance from the origin.

You are presented with the following data. Each data point $\vec{d_i}^T = [x_i\ y_i]^T$ has the corresponding label $l_i \in \{-1,1\}$.

x_i	y_i	l_i
-2	1	-1
-1	1	1
1	1	1
2	1	-1

Labels for data you are classifying

(a) (6 points) You want to build a model to understand the data. You first consider a linear model, i.e. you want to find $\alpha, \beta, \gamma \in \mathbb{R}$ such that $l_i \approx \alpha x_i + \beta y_i + \gamma$.

Set up a least squares problem to solve for α, β and γ. If this problem is solvable, solve it, i.e. find the best values for α, β, γ. If it is not solvable, justify why.
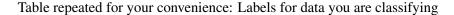
Solution: Rewriting the equations $\alpha x_i + \beta y_i + \gamma \approx l_i$ for $i = 1,2,3,4$ in matrix form gives:
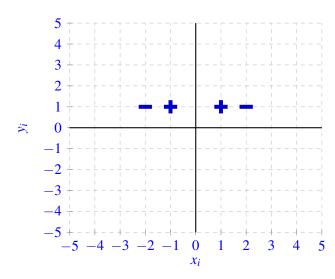
$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \approx \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$
$$\mathbf{A}\vec{x} \approx \vec{b}$$

The least squares solution is $\hat{\vec{x}} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \vec{b}$. The solution only exists when the matrix $\mathbf{A}^T \mathbf{A}$ is invertible, and an equivalent condition is when all the columns of $\mathbf{A}$ are linearly independent. We see that the second and third columns of $\mathbf{A}$ are linearly dependent, so the problem is **not** solvable.

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(b) (3 points) **Plot** the data points in the plot below with axes (x_i, y_i). **Is there a straight line such that the data points with a $+1$ label are on one side and data points with a -1 label are on the other side? Answer yes or no, and if yes, draw the line.**

x_i	y_i	l_i
-2	1	-1
-1	1	1
1	1	1
2	1	-1

Table repeated for your convenience: Labels for data you are classifying

Solution:

The answer is no. As can be seen from the plot, the points all lie on the line $y_i = 1$, so there is no line that is able to separate the points based on their label.

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(c) (6 points) You now consider a model with a quadratic term: $l_i \approx \alpha x_i + \beta x_i{}^2$ with $\alpha, \beta \in \mathbb{R}$. *Read the equation carefully!*

Set up a least squares problem to fit the model to the data. If this problem is solvable, solve it, i.e, find the best values for α, β. If it is not solvable, justify why.

x_i	y_i	l_i
-2	1	-1
-1	1	1
1	1	1
2	1	-1

Table repeated for your convenience: Labels for data you are classifying

Solution: Rewriting the equations $\alpha x_i + \beta x_i^2 \approx l_i$ for $i = 1, 2, 3, 4$ in matrix form gives:
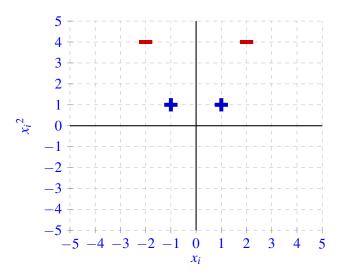
$$
\begin{bmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \\ x_3 & x_3^2 \\ x_4 & x_4^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 1 \\ 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \approx \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}
$$
$$
\mathbf{A}\vec{x} \approx \vec{b}
$$

The least squares solution is $\hat{\vec{x}} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \vec{b}$. The solution only exists when the matrix $\mathbf{A}^T \mathbf{A}$ is invertible, and an equivalent condition is when all the columns of $\mathbf{A}$ are linearly independent. We see that the first and second columns of $\mathbf{A}$ are linearly independent, so the problem is solvable.

We can solve for $\hat{\vec{x}} = \begin{bmatrix} \alpha & \beta \end{bmatrix}^T = \begin{bmatrix} 0 & \frac{-3}{17} \end{bmatrix}^T$.

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(d) (3 points) **Plot** the data points in the plot below with axes (x_i, x_i^2). **Is there a straight line such that the data points with a $+1$ label are on one side and data points with a -1 label are on the other side? Answer yes or no, and if yes, draw the line.**

x_i	y_i	l_i
-2	1	-1
-1	1	1
1	1	1
2	1	-1

Table repeated for your convenience: Labels for data you are classifying

Solution:

The answer is yes. A line $x_i^2 = u$ where $1 < u < 4$ would separate the data points based on their labels. It is important to note that solving the least squares problem considered so far would not yield that line because the problem so far considers only lines that pass through the origin.

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(e) (4 points) Finally you consider the model: $l_i \approx \alpha x_i + \beta x_i^2 + \gamma$, where $\alpha, \beta, \gamma \in \mathbb{R}$. Independent of the work you have done so far, **would you expect this model or the model in part (c) (i.e. $l_i \approx \alpha x_i + \beta x_i^2$) to have a smaller error in fitting the data? Explain why.**

x_i	y_i	l_i
-2	1	-1
-1	1	1
1	1	1
2	1	-1

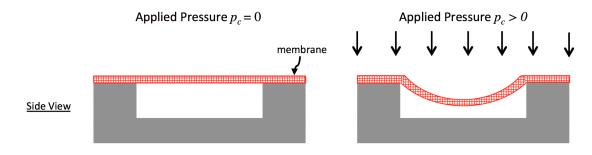Table repeated for your convenience: Labels for data you are classifying

Solution: We expect the model in part (e) to have a smaller error because there are more degrees of freedom. The model in part (c) only considers lines passing through the origin, while the model in part (e) considers all lines. With the model in part (e) we are able to obtain a line $x_i^2 = u$ where $1 < u < 4$ that would separate the data points based on their labels, unlike the model in part (c).

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5. **Putting on the Pressure: Build your own InstantPot (27 points)** (All subparts of this problem can be solved independently.)

Prof. Ranade tells Prof. Boser about her great experience with her automatic pressure cooker, and they decide to try and build one together. The design of the pressure cooker uses a pressure sensor and heating element. Whenever the pressure is below a set target value, an electronic circuit turns on the heating element.
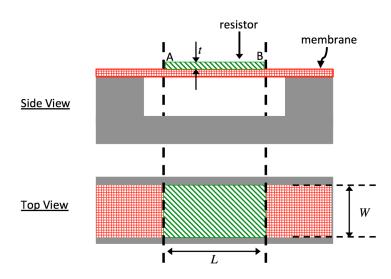
Pressure Sensor Resistance

The first step is designing a pressure sensor; the figure below shows your design. As pressure p_c is applied, the flexible membrane stretches.

Applied Pressure $p_c = 0$ Applied Pressure $p_c > 0$

membrane

Side View

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(a) (4 points) Now you attach a resistor layer R_p with resistivity $\rho = 0.1$ Ωm, width W, length L, and thickness t to the pressure sensor membrane, as illustrated in the figure below. When the pressure $p_c = 0kPa$ (i.e. there is no applied pressure), $W = 1mm$, $L = 1cm$, $t = 100\mu m = 100 \times 10^{-6}m$. **Calculate the value of R_p when there is no applied pressure.** Note that direction of current flow in the resistor is from A to B as marked in the diagram. Show all your work.

Solution:

Resistance $R_p = \frac{\rho \times L}{A} = \frac{\rho \times L}{W \times t} = \frac{=0.1\Omega m \times 0.01m}{0.001m \times 100 \times 10^{-6}m} = 10k\Omega$.

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(b) (5 points) When pressure is applied, the length L of the resistor changes and is a function of applied pressure p_c, and is given by

$$L(p_c) = L_0 + \beta p_c,$$

where L_0 is the length of the resistor with no pressure applied, and β is a constant. As a result, the value of resistance R_p changes from its nominal value R_{p0} (the value of R_p with no pressure applied) **Derive an expression for $R_p(p_c)$ as a function of resistivity ρ, width W, thickness t, nominal length L_0, constant β, and applied pressure p_c.**

Note: The width and thickness of the resistor will also change with applied pressure. However, we ignore this to keep the math simple.

Solution:

Resistance $R_p = \frac{\rho \times L(p_c)}{A} = \frac{\rho L}{W \times t} = \frac{\rho \times (L_0 + \beta p_c)}{W \times t}$

Extra space for scratch work.

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(c) **Pressure Sensor Circuit** (6 points)

Now assume that the resistance R_p is a function of applied pressure p_c according to the relationship $R_p(p_c) = R_o \times \frac{p_c}{p_{ref}}$ where $R_o = 1\text{k}\Omega$ and $p_{ref} = 100\text{kPa}$.

To complete our sensor circuit, we would like to generate a voltage V_p that is a function of the pressure p_c. **Complete the circuit below so that the output voltage V_p depends on the pressure p_c as:**

$$V_p(p_c) = -V_o \times \frac{p_c}{p_{ref}}, \text{ where } V_o = 1V.$$

- **You may add at most one ideal voltage source and one additional resistor to the circuit, but you must calculate their values and mark them in the diagram.**
- **Mark the positive and negative inputs of the operational amplifier with "+" and "-" symbols, respectively, in the boxes provided.**
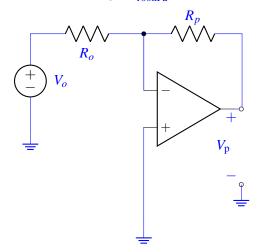
You may assume that the operational amplifier is ideal.

Solution:

We can combine the relationships $R_p(p_c) = R_o \times \frac{p_c}{p_{ref}}$ and $V_p(p_c) = -V_o \times \frac{p_c}{p_{ref}}$ to get the relationship:

$V_p(p_c) = -V_o \times \frac{R_p(p_c)}{R_o}$.

We can then use an inverting amplifier op-amp configuration (see Discussion 10B) to realize the circuit.

$R_o = 1\text{k}\Omega$ and $R_p = \frac{1\text{k}\Omega}{100\text{kPa}} \times p_c$.

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(d) **Resistive Heating Element** (4 points)

To heat the pressure cooker, you use a heating element with resistance R_{heat}. **Calculate the value of R_{heat} such that the power dissipated is $P_{heat} = 1000W$ with $V_{heat} = 100V$ applied across the heating element.**

Solution:

$P_{heat} = V_{heat} I_{heat} = V_{heat} \times \frac{V_{heat}}{R_{heat}}$. Therefore, $R_{heat} = \frac{V_{heat}^2}{P_{heat}} = \frac{(100V)^2}{1000W} = 10\Omega$.
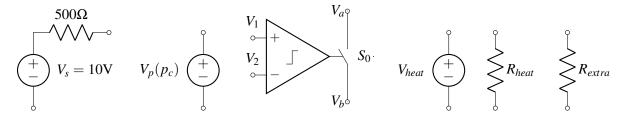
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(e) **Pressure Regulation** (8 points)

You are finally ready to complete the design of your pressure cooker.

Using the circuit elements below, make a circuit that will turn the heater to on (i.e. current flowing through R_{heat}) when the pressure is less than 500 kPa, and off (i.e. no current flowing through R_{heat}) when the pressure is greater than 500 kPa.
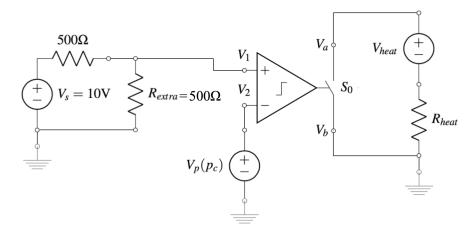
The elements are:

- A voltage source $V_s = 10\text{V}$ with a Thevenin resistance of 500Ω.
- A voltage source $V_p(p_c) = V_o \times \frac{p_c}{p_{ref}}$, with $V_o = 1\text{V}$ and $p_{ref} = 100\text{kPa}$. (This is a voltage source whose voltage is a function of pressure p_c, unrelated to any previous parts of the question.)
- A comparator that controls switch S_0. The switch is normally opened (i.e. an open circuit between nodes V_a and V_b), and is closed only when $V_1 > V_2$ (i.e. a short circuit between nodes V_a and V_b).
- The heater supply ($V_{heat} = 100\text{V}$).
- The heater resistor R_{heat}.
- One additional resistor R_{extra}. If you use this resistor you must calculate and note its value on your circuit diagram.
- You may assume you have access to a ground node.

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Solution:

We first must compute the value of $V_p(500\text{kPa})$. We can compute $V_p(500\text{kPa}) = 5V$. This is the voltage that we want to compare against the output of V_p in order to make sure the heater is on when the pressure p_c is less than 500kPa and off when the pressure p_c is greater than 500kPa. We can make a 5V reference by using R_{extra} to make a voltage divider with the voltage source V_s. By making R_{extra} equal to the source's Thevenin resistance the output of the voltage divider is 5V. We connect this to the positive input of the comparator. We then connect the source $V_p(p_c)$ to the negative input of the comparator.
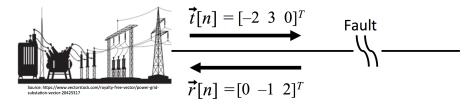
We then connect the voltage source V_{heat} in series with the resistor R_{heat} so that when S_0 is closed (i.e. when the pressure p_c is less than 500kPa) power will be delivered to R_{heat}.

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6. **Finding faults with PG&E (16 points)** (All subparts of this problem can be solved independently.)

PG&E has been having problems with the grid, and needs to more accurately locate faults in their transmission lines. EECS 16A students decide to use their new design skills to help.

All the transmission lines are connected to substations. To find a fault, a substation sends a signal $\vec{t}$ down a transmission line. If the line has a break in it, then a reflection occurs and the substation receives back a signal $\vec{r}$.
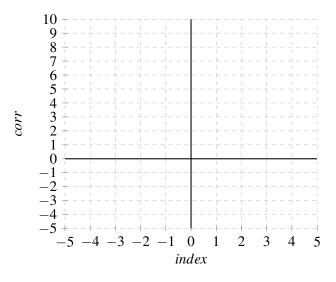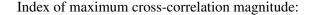

$$\vec{t}[n] = [-2\ 3\ 0]^T$$ Fault

$$\vec{r}[n] = [0\ -1\ 2]^T$$

Source: https://www.vectorstock.com/royalty-free-vector/power-grid-substation-vector-20425317

(a) (6 points) Assume that a substation sends the signal $\vec{t}$ and receives back a signal $\vec{r}$. The received signal $\vec{r}$ is a delayed, scaled version of $\vec{t}$ with added noise.

$$\vec{t}[n] = \begin{bmatrix} -2 & 3 & 0 \end{bmatrix}^T \tag{10}$$

$$\vec{r}[n] = \begin{bmatrix} 0 & -1 & 2 \end{bmatrix}^T \tag{11}$$

Use the axes below to plot the cross-correlation $corr_{\vec{r}}(\vec{t})$, and use this to identify the index corresponding to the peak (maximum magnitude) of the cross-correlation.

Index of maximum cross-correlation magnitude:

Solution: Cross Correlation given by:

$$corr_{\vec{r}}(\vec{t}) = \begin{bmatrix} 0 & 0 & -3 & 8 & -4 \end{bmatrix} \tag{12}$$

Where argmax is index 1.

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(b) (10 points) Now the substations send signals along the transmission lines to send information about the state of the power grid. **Each substation has a unique code that it uses to transmit its information, $\vec{s}_1$, $\vec{s}_2$, $\vec{s}_3$, and $\vec{s}_4$. You receive the signal $\vec{r}[n] = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \end{bmatrix}^T$ and you know it contains signals from two different substations.**

Since the locations of the various substations are known, we can compute the delay with which the signals are received. The delays corresponding to the max correlation of $\vec{s}_1$, $\vec{s}_2$, $\vec{s}_3$, and $\vec{s}_4$ are: 1 unit, 2 units, 1 unit, and 2 units respectively. **We have provided shifted versions of the signals $(\vec{u}_1[n], \vec{u}_2[n], \vec{u}_3[n], \vec{u}_4[n])$** that correspond to these distances, and included appropriate zeros to the signals **to make your calculations easier.**

$$\vec{s}_1[n] = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \xrightarrow{\text{delayed by 1}} \vec{u}_1[n] = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \end{bmatrix}^T, \tag{13}$$

$$\vec{s}_2[n] = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T \xrightarrow{\text{delayed by 2}} \vec{u}_2[n] = \begin{bmatrix} 0 & 0 & 1 & -1 & 1 \end{bmatrix}^T, \tag{14}$$

$$\vec{s}_3[n] = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T \xrightarrow{\text{delayed by 1}} \vec{u}_3[n] = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 \end{bmatrix}^T, \tag{15}$$

$$\vec{s}_4[n] = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}^T \xrightarrow{\text{delayed by 2}} \vec{u}_4[n] = \begin{bmatrix} 0 & 0 & -1 & -1 & 1 \end{bmatrix}^T. \tag{16}$$

Determine which two unique signals are contained in the received signal $\vec{r}[n]$. What are the weights on the two signals? Show all of your work.

Some calculations that might be useful:

$< \vec{r}[n], \vec{u}_1[n] >= 5$	$< \vec{u}_1[n], \vec{u}_2[n] >= 0$	$< \vec{u}_2[n], \vec{u}_3[n] >= 2$	$< \vec{u}_3[n], \vec{u}_4[n] >= 0$
$< \vec{r}[n], \vec{u}_2[n] >= 0$	$< \vec{u}_1[n], \vec{u}_3[n] >= 1$	$< \vec{u}_2[n], \vec{u}_4[n] >= 1$	
$< \vec{r}[n], \vec{u}_3[n] >= 1$	$< \vec{u}_1[n], \vec{u}_4[n] >= -2$		
$< \vec{r}[n], \vec{u}_4[n] >= -2$			

This might also help:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{17}$$

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Solution: Start with $\vec{u}_1$ in our first iteration of OMP because it has the highest value inner product. As done in discussion, we can find the next error vector:

$$\vec{e}_1 = \vec{r} - proj_{\vec{u}_1}(\vec{r}) \tag{18}$$

$$= \vec{r} - \frac{<\vec{r}, \vec{u}_1>}{<\vec{u}_1, \vec{u}_1>} \vec{u}_1 \tag{19}$$

$$= \begin{bmatrix} 1 & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} & 1 \end{bmatrix}^T \tag{20}$$

Then we can compute the new inner products as:

$$<\vec{e}_1, \vec{u}_i> = <\vec{u}_i, \vec{r} - \frac{<\vec{r}, \vec{u}_1>}{<\vec{u}_1, \vec{u}_1>} \vec{u}_1> \tag{21}$$

$$<\vec{e}_1, \vec{u}_2> = 0 \tag{22}$$

$$<\vec{e}_1, \vec{u}_3> = -\frac{2}{3} \tag{23}$$

$$<\vec{e}_1, \vec{u}_4> = \frac{4}{3} \tag{24}$$

$$\tag{25}$$

Since the highest inner product is with $\vec{u}_4$ we now solve for x_1 and x_4, i.e. the weights of $\vec{u}_1$ and $\vec{u}_4$ using least squares:

Let $A = \begin{bmatrix} \vec{u}_1 & \vec{u}_4 \end{bmatrix}$.

$$\hat{x} = \begin{bmatrix} x_1 & x_4 \end{bmatrix}^T = (A^T A)^{-1} A^T \vec{r} \tag{26}$$

$$= \left(\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & -1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \tag{27}$$

$$= \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ -2 \end{bmatrix} \tag{28}$$
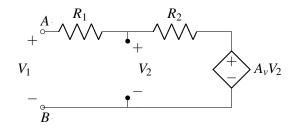
$$= \frac{1}{5} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} \tag{29}$$

$$= \frac{1}{5} \begin{bmatrix} 11 \\ 4 \end{bmatrix} \tag{30}$$

So we have $x_1 = 2.2$ and $x_4 = 0.8$.

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7. **Fun With Circuits (14 points)** (All subparts of this problem can be solved independently.)

In his spare time Professor Boser invents new circuits. The circuit schematic below shows his latest creation that uses a voltage controlled voltage source.
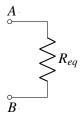

(a) **Equivalent Resistance** (8 points)

Analyze the circuit above and model it with an equivalent resistance R_{eq} as illustrated below. The I/V curves of the original and equivalent circuit should be identical. Use the following values: $A_v = 3$, $R_1 = 1\ k\Omega$, and $R_2 = 4\ k\Omega$. Show your calculations.

Recall that a resistor is just a model of the IV dependence of a circuit element; you may get a positive or a negative answer for your equivalent resistance.

Hint: Apply a test voltage (or current) and compute the resulting current (or voltage).

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Solution:

Let us apply a test voltage across the terminals A and B such that $V_1 = V_{test}$.

Let us also define I_{R_1} and I_{R_2} as the current that flows through R_1 and R_2, respectively. Let us also define the currents as both going from left to right.

Therefore, we know from KCL that $I_{R_1} = I_{R_2}$.

We also know that $I_{R_1} = \frac{V_1 - V_2}{R_1}$ and that $I_{R_2} = \frac{V_2 - A_v V_2}{R_2}$.

Therefore, $\frac{V_1 - V_2}{R_1} = \frac{V_2 - A_v V_2}{R_2}$.

Simplifying for V_2 we get $V_2 = \frac{V_1 R_2}{R_1 - A_v R_1 + R_2}$.

We can then back substitute this result into I_{R_1} to find I_{R_1} as a function of V_1, R_1, R_2, and A_v, and we find $I_{R_1} = \frac{V_1(1 - A_v)}{R_1(1 - A_v) + R_2}$.

We can then solve for $R_{eq} = \frac{V_1}{I_{R_1}} = R_1 + \frac{R_2}{1 - A_v}$.

Plugging in $R_1 = 1\text{k}\Omega$, $R_2 = 4\text{k}\Omega$, and $A_v = 3$, we find that $R_{eq} = -1\text{k}\Omega$.

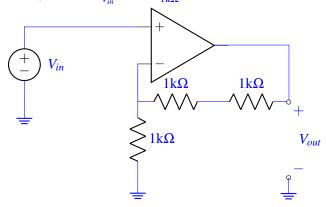PRINT your name and student ID: _____

(b) **Amplifier Design** (6 points) This part is independent of the previous part.

Design a circuit which implements a voltage controlled voltage source with gain $A_v = 3$, i.e. $V_{out} = 3V_{in}$. Use the ideal operational amplifier shown below and up to three additional $1\,k\Omega$ resistors. Label V_{in} and V_{out}.

Solution:

We use a non-inverting operational amplifier configuration (see Discussion 10B) to make an amplifier with gain $A_v = \frac{V_{out}}{V_{in}} = 1 + \frac{2k\Omega}{1k\Omega} = 3$.

Note that in part (a) the circuit uses an ideal voltage controlled voltage source. In part (b) we use an operational amplifier to actually synthesize the voltage controlled voltage source. Conclusion: amplifiers can be used to synthesize unusual circuit behavior (i.e. negative resistance).

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8. **Projections and eigenvectors** (32 points) (All subparts of this problem can be solved independently.)

Consider two n-dimensional vectors $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$. Consider the matrix $\mathbf{M} = \vec{x}\,\vec{y}^T$. **Note the order of the multiplication, this is distinct from** $<\vec{x}, \vec{y}> = \vec{x}^T \vec{y}$.

(a) (2 points) What are the dimensions of matrix $\mathbf{M}$?

Solution: $\mathbf{M}$ is an $n \times n$ matrix. This is different from the inner product, which would yield a 1×1 matrix which we treat as a scalar.

(b) (6 points) Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. **Find the eigenvalues and eigenvectors of M.** Show your work.

Solution: Multiplying, we see that
$$\mathbf{M} = \begin{bmatrix} 4 & 5 \\ 8 & 10 \end{bmatrix}.$$

Computing the determinant of $\mathbf{M} - \lambda \mathbf{I}$, we obtain

$$|\mathbf{M} - \lambda \mathbf{I}| = (4 - \lambda)(10 - \lambda) - 40.$$

Setting this quantity to zero, we can solve to obtain

$$\lambda^2 - 14\lambda = 0 \implies \lambda = 0 \text{ or } 14.$$

Thus, looking at the null spaces of $\mathbf{M}$ and $\mathbf{M} - 14\mathbf{I}$, we obtain the eigenvalue-eigenvector pairs

$$\left(0, \begin{bmatrix} 5 \\ -4 \end{bmatrix} \right) \text{ and } \left(14, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right).$$

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(c) (6 points) Now let $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$. Let $\mathbf{M} = \vec{x}\,\vec{y}^T$. **Find the projection of $\vec{x}$ onto the columnspace of M.** Show all your work. *Hint: Write out the columns of* M.

Solution: Let the components of $\vec{y}$ be

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Then we see that

$$\mathbf{M} = \vec{x}\vec{y}^T = \begin{bmatrix} | & | & & | \\ y_1\vec{x} & y_2\vec{x} & \cdots & y_n\vec{x} \\ | & | & & | \end{bmatrix}.$$

So the column space of **M** is just $\mathrm{span}(\vec{x})$. Thus, $\vec{x}$ already lies in this subspace, so the projection is just $\vec{x}$ itself.

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(d) (8 points) $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$. Let $\mathbf{M} = \vec{x}\,\vec{y}^T$. Let $\vec{z} \in \mathbb{R}^n$ be a vector such that $< \vec{z}, \vec{y} >= 0$. **Prove that** $\vec{z} \in Null(\mathbf{M})$**.** You must prove this from first principles. No theorems can be used in the proof. Show all your work.

Solution: By the definition of inner products, $\vec{y}^T \vec{z} = 0$. Thus,

$$\mathbf{M}\vec{z} = \vec{x}\vec{y}^T \vec{z} = 0\vec{x} = \vec{0},$$

so $\vec{z} \in \text{Null}(\mathbf{M})$, as desired.

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(e) (10 points) $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$. Let $\mathbf{M} = \vec{x}\,\vec{y}^T$. Further, assume that $<\vec{x}, \vec{y}> \neq 0$. **Find an eigenvector of M corresponding to a non-zero eigenvalue.** Show all your work and justify your answer.

Solution: From looking for patterns in part (a), we conjecture that $\vec{x}$ is an eigenvector of $\mathbf{M}$. To verify this conjecture, we observe that

$$\mathbf{M}\vec{x} = \vec{x}\vec{y}^T\vec{x} = (\vec{y}^T\vec{x})\vec{x} = <\vec{x}, \vec{y}>\vec{x},$$

so $\vec{x}$ is an eigenvector of $\mathbf{M}$ with eigenvalue $\lambda = <\vec{x}, \vec{y}> \neq 0$, as desired. Thus, $\vec{x}$ is such an eigenvector.

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9. **Electronic Level (24 points)** (All subparts of this problem can be solved independently.)

A one-wheel scooter (as shown below) is fun to ride, but unfortunately quite expensive to buy. Inspired by the success of the pressure cooker, you decide to build your own.

One challenge of building a one-wheel scooter is designing a circuit that measures the angle α of the riding platform, as shown above. You are considering the design of an angle sensor consisting of two capacitors C_a and C_b whose values depend on the angle α (measured in degrees) as follows:

$$C_a(\alpha) = C_o \left(100 + \frac{\alpha}{\alpha_{ref}} \right), \tag{31}$$

$$C_b(\alpha) = C_o \left(100 - \frac{\alpha}{\alpha_{ref}} \right), \tag{32}$$

where $\alpha_{ref} \neq 0$ and C_o are properties of the sensor.

The circuit in Fig. 9.1 produces a voltage $V_{out}(k)$ at time k that depends on α through the capacitances $C_a(\alpha), C_b(\alpha)$. The timing diagram (Fig. 9.2) shows the state of the switches as a function of time k. **At time $k = 0$ all of the capacitors are completely discharged, meaning that $\mathbf{V_{out}(0) = 0V}$.**
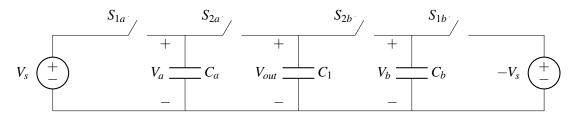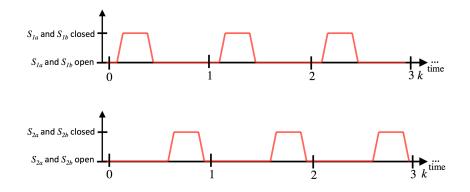

Figure 9.1: The circuit for the electronic level

Figure 9.2: The timing diagram for the circuit
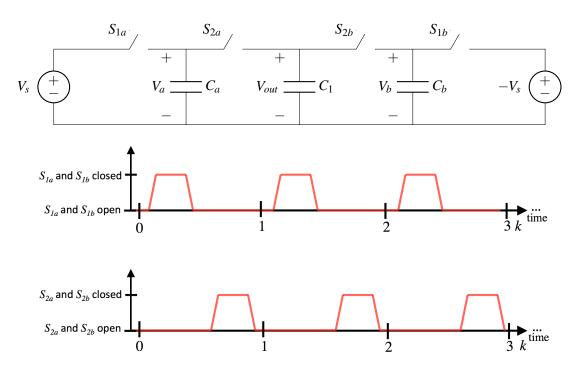
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(a) (5 points) We define $Q_a = C_aV_a$, $Q_b = C_bV_b$ and $Q_1 = C_1V_{out}$, per the labels in Figure 9.1. **Find an expression for the total charge on the capacitors $Q_{tot}(k) = Q_a + Q_b + Q_1$ at time $k = 1$ as a function of α, α_{ref} C_o, C_1, and V_s.** At time $k = 1$, all switches have been closed and opened once and are now in the open state. Show all your work.

Solution:

Because all of the capacitors are discharged at k=0, the only additional charge at k=1 is due to the charge on C_a and C_b.

We can write

$$Q_{tot}(1) = Q_a + Q_b = C_aV_s - C_bV_s = 2V_sC_o\frac{\alpha}{\alpha_{ref}}. \tag{33}$$

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The same circuit and timing diagram from earlier, reproduced for your convenience.

(b) (5 points) **Calculate $V_{out}(1)$ as a function of $Q_{tot}(1)$, C_o, and C_1.** Show your work.

Solution:

$$V_{out}(1) = \frac{Q_{tot}(1)}{C_a + C_b + C_1} = \frac{Q_{tot}(1)}{200C_o + C_1}. \tag{34}$$

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(c) (8 points) **Find an expression for $V_{out}(k)$ as a function of $V_{out}(k-1)$, α, α_{ref}, C_o, C_1, and V_s.** It is important to have an exact formula. Vague answers will receive no credit. Show your work. *Hint: You might try to first find $V_{out}(2)$ as a function of $V_{out}(1)$.*
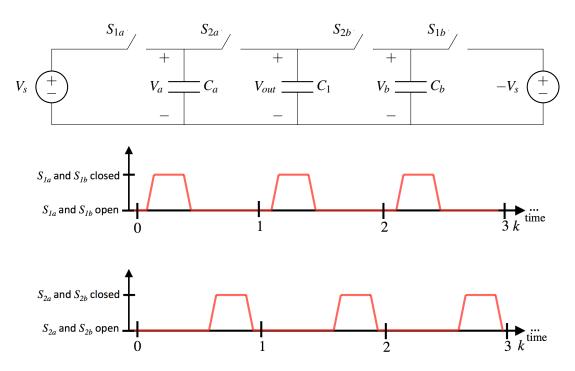
Solution:

Let us define the total charge on C_1 at the end of timestep $k-1$ as $Q_1(k-1)$. During timestep k, there is an additional charge of $V_s(C_a - C_b)$ that is added to the system because of the fact that C_a and C_b are charged to V_s and $-V_s$, respectively.

Therefore, we can write $Q_{tot}(k) = Q_1(k-1) + V_s(C_a - C_b)$.

We can then use the relationship $Q_1(k-1) = C_1 V_{out}(k-1)$ and write:

$$V_{out}(k) = \frac{Q_{tot}(k)}{C_a + C_b + C_1} \tag{35}$$

$$= \frac{C_1 V_{out}(k-1) + V_s(C_a - C_b)}{C_a + C_b + C_1} = \frac{C_1 V_{out}(k-1) + 2V_s C_o \frac{\alpha}{\alpha ref}}{200 C_o + C_1}. \tag{36}$$

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The same circuit and timing diagram from earlier, reproduced for your convenience.

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(d) (6 points) It turns out that the behavior of this circuit is exactly like the linear control systems you modeled in Module 1, and can be analyzed similarly.

Assume
$$V_{out}(k) = \gamma V_{out}(k-1) + \beta.$$

Both $\gamma, \beta \in \mathbb{R}$, and $0 < \gamma < 1$. Recall $V_{out}(0) = 0$. **What does V_{out} converge to as $k \to \infty$, i.e., what is $V_{out}(\infty)$?** Your answer should be in terms of β, γ and numbers. Show your work. (*Hint: The infinite series $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ if $0 < x < 1$.*)

Solution:

We can see from part (c) that we have an expression of the form $V_{out}(k) = \gamma V_{out}(k-1) + \beta$. Let us first solve for $V_{out}(1) = \gamma V_{out}(0) + \beta = \beta$.

Let us then solve for $V_{out}(2), V_{out}(3), V_{out}(4)$, etc.

$V_{out}(2) = \gamma V_{out}(1) + \beta = \gamma\beta + \beta,$

$V_{out}(3) = \gamma V_{out}(2) + \beta = \gamma^2\beta + \gamma\beta + \beta,$

$V_{out}(4) = \gamma V_{out}(3) + \beta = \gamma^3\beta + \gamma^2\beta + \gamma\beta + \beta,$

...

$V_{out}(k) = \gamma^{k-1}\beta + \gamma^{k-2}\beta + ... + \gamma\beta + \beta = \beta(\gamma^{k-1} + \gamma^{k-2} + ... + \gamma + 1).$

We can then solve for $V_{out}(\infty) = \frac{\beta}{1-\gamma}$.

The next part is not required for the solution but is just for those of you who are curious. You have $\gamma = \frac{C_1}{200C_o + C_1}$ and $\beta = \frac{2V_sC_o\frac{\alpha}{\alpha_{ref}}}{200C_o + C_1}$.

If you combine this with your answer from the previous part you will get: $V_{out}(\infty) = \frac{V_s\alpha}{100\alpha_{ref}}$. So as time goes to infinity, the circuit converges to a function of only α and does not depend on time i.e. the number of times the switches have been opened and closed!

A side benefit of this is that even if some "noise" charge gets onto the capacitors, over time the constant decay due to the $\gamma < 1$ will ensure that this noise is removed and does not affect the final value that the circuit converges to. As k increases, $V_{out}(k)$ converges to a constant value even if noise has been added in the past.

The idea can also be understood from the effect of charge sharing. At each time step, C_a and C_b are charged to V_s and $-V_s$, respectively, and then share this charge with C_1 (along with whatever charge is already on C_1). Even if there is noise introduced on $V_{out}(k)$ (i.e. extra charge on C_1), this will get shared with C_a and C_b, and as C_a and C_b are charged to V_s and $-V_s$ in the future, this extra charge will bleed away.