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11:30am-2:30pm

EECS 16A: SPRING 2015—FINAL

Important notes: Please read every question carefully and completely – the setup may or may not be the same as what you have seen before. Also, be sure to show your work since that is the only way we can potentially give you partial credit.

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Total: _____/ 131

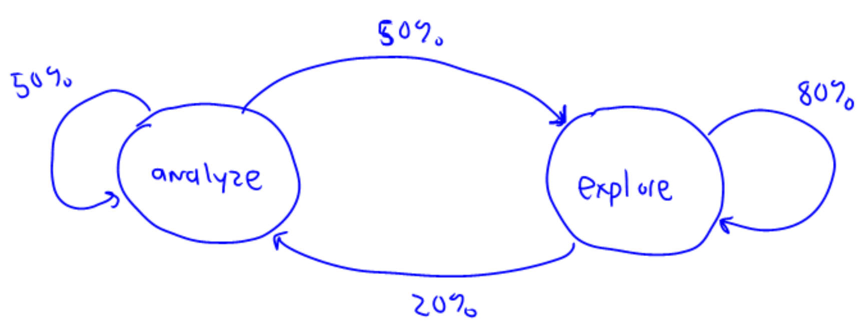
PROBLEM 1. Robot Explorers I (14 pts + BONUS 6 pts)

In this series of two problems we'll look at some key aspects of building an autonomous system made out of a group (swarm) of robots meant to explore unknown spaces. First, let's think about the design of the controller that coordinates all of the robots. We want to design an algorithm that lets the robots wander around an area and generally map the interesting objects in it. However, whenever one of the robots does discover an interesting object, we want to make sure it has the opportunity to analyze it carefully.

Let's look at one such possible algorithm to achieve this goal. At each time step n, each robot can be in one of two modes – *explore*, or *analyze*. Depending upon which mode it is in, the robot can decide whether to stay in the same mode, or switch to the other mode. In particular, if we let n be a time index, let's assume that out of all of the robots that are in the *explore* mode at time n, 20% of them will detect an interesting object and switch to the *analyze* mode at time n+1.

You have no design control over the rate at which interesting objects will be found. However, you can design the parameters of the *analyze* mode. You decide that for the robots that are in the *analyze* mode at time n, you will command 50% of them to stop analyzing and switch to the *explore* mode at time n+1.

a) **(2 pts)** Draw a graph that models this setup. The nodes in the graph should be *analyze* and *explore*, and the arcs between the nodes should represent either the percentage of robots that stay in the same state or the percentage of robots that switch to the other state.



b) **(4 pts)** Now let's set up a vector $x(n) = [x_{explore}(n)\ x_{analyze}(n)]^T$, where $x_{analyze}$ represents the fraction of robots in *analyze* mode, and $x_{explore}$ represents the fraction of robots in the *explore* mode. Write a matrix $A_{transition}$ that you could multiply $x(n)$ by to predict $x(n+1)$ (i.e., so that $x(n+1) = A_{transition} * x(n)$).

$$A_{transition} = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$$

c) **(8 pts)** With the parameters given in part (a), will this algorithm result in a steady state for number of robots in *explore* mode vs. the number in *analyze* mode? If so, in steady state, what will be the percentage of robots in each state?

Look to see if $A_{transition}$ has a eigenvector with eigenvalue of 1:

$$\det(A - \lambda I) = (0.8-\lambda)(0.5-\lambda) - 0.2 \cdot 0.5 = 0$$

$$\lambda^2 - 1.3\lambda + 0.4 - 0.1 = 0$$

$$\lambda = \frac{1.3 \pm \sqrt{1.3^2 - 1.2}}{2}$$

$$\hookrightarrow \lambda_1 = 1, \quad \lambda_2 = 0.3$$

There is a steady state, so let's find the associated eigenvector:

$$\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \longrightarrow \begin{aligned} 0.8v_1 + 0.5v_2 &= v_1 \\ 0.2v_1 &= 0.5v_2 \\ v_1 &= 2.5v_2 \end{aligned}$$

Also know that $v_1 + v_2 = 1$, so steady state is: $\begin{bmatrix} x_{explore} \\ x_{analyze} \end{bmatrix} = \begin{bmatrix} 2.5/3.5 \\ 1/3.5 \end{bmatrix}$

d) **(BONUS: 6 pts)** If you wanted the steady state to be half of the robots exploring and the other half analyzing, what percentage of the robots in the *analyze* state should you command to switch to the *explore* state at any particular time step *n*?

Want $\begin{bmatrix} 0.8 & P_{switch} \\ 0.2 & 1-P_{switch} \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

$$0.4 + 0.5 P_{switch} = 0.5$$

$$\boxed{P_{switch} = 0.2}$$

PROBLEM 2. Robot Explorers 2 (23 pts)

This problem explores the aging of a single robot, Wall-E, that has been successfully exploring post-"junk apocalypse" Earth for many years. Wall-E was programmed to take in two control signals as commands: Δx and Δy. When everything operates as originally intended, these commands as a vector $[\Delta x\ \Delta y]^T$ would move Wall-E by $[\Delta x\ \Delta y]^T$ away from its current position.

a) **(3 pts)** After so many years of exploration, the wear and tear on the motors Wall-E uses to move is now starting to slightly affect their behavior, and Wall-E's movements become more erratic in a particular way. Specifically, when given the input vector control $[\Delta x\ \Delta y]^T$, instead of moving by $[\Delta x\ \Delta y]^T$, Wall-E moves by $A*[\Delta x\ \Delta y]^T$, where $A=\begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}$. Under this new model, what control input - i.e. what $[\Delta x\ \Delta y]^T$ – would you have to apply input to get Wall-E to move by $[1\ 2]^T$ relative to its current position? Note that you can leave your answer in the form of an equation – i.e., you don't need to provide any numerical results.

In order to move by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, we need to apply $A^{-1} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

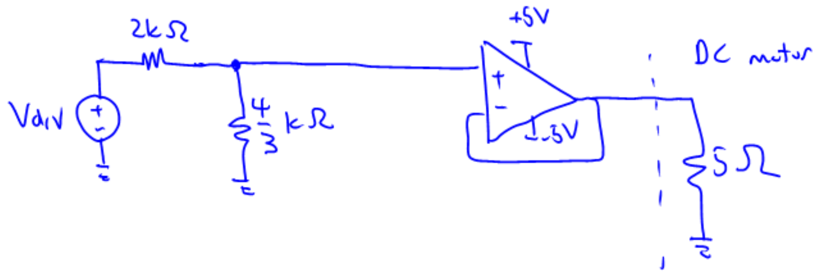
b) **(8 pts)** In order to actually move, Wall-E (like most robots) uses what's known as a "DC motor". We won't really go in to how DC motors actually work, so for the rest of this problem, we will model each DC motor as a resistor with value $R_{in} = 5\Omega$.

If we have a voltage source V_{drv} with a source resistance of $2k\Omega$ that ranges from -2.5V to 2.5V, design a circuit that will drive the DC motor with a voltage that is proportional to V_{drv}, but that ranges from -1V to 1V. You can assume that one side of the motor's input is connected to ground, and you can use any number of op-amps and resistors for your circuit. You can further assume that you have +/-5V voltage sources that you can use as the power supplies for your op-amps.

Block diagram:

$$V_{drv} \longrightarrow \boxed{k} \longrightarrow V_{out}$$

$$k = \frac{1V}{2.5V} = 0.4$$



To get $k = 0.4$: $\dfrac{x}{2+x} = 0.4$

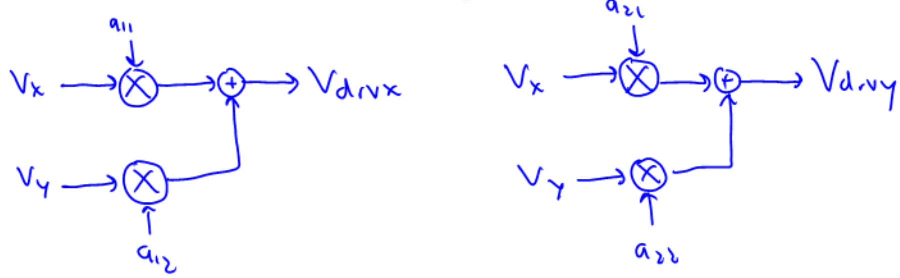
$$x = 0.8 + 0.4x$$

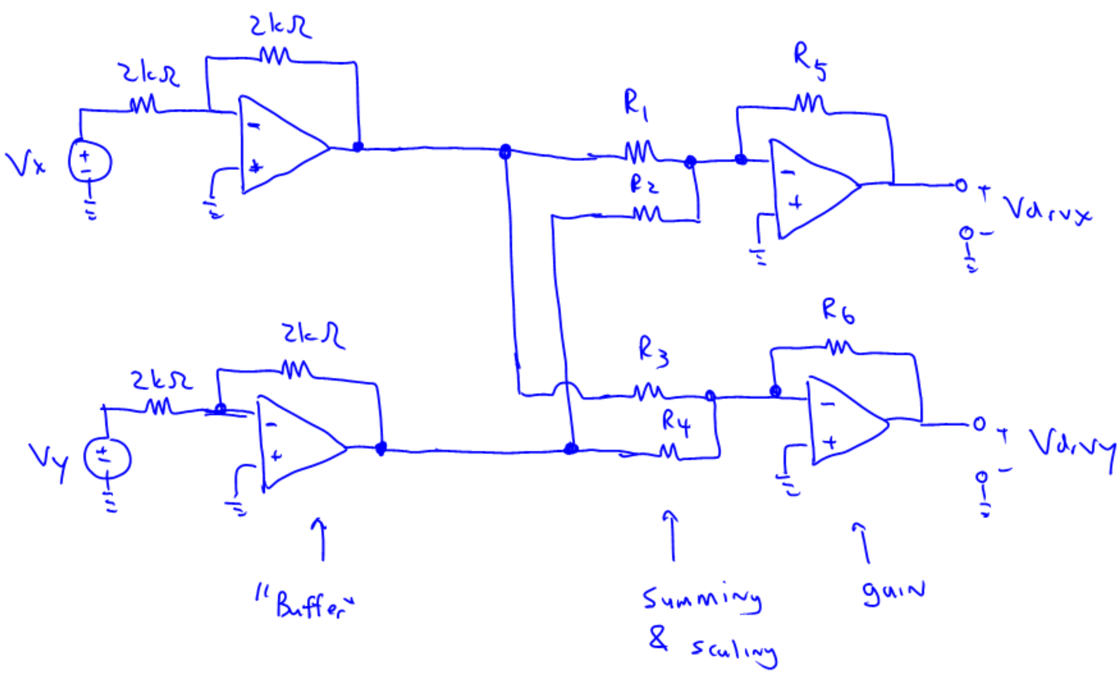
$$x = 4/3$$

c) **(12 pts)** As we saw in part (b), wear and tear on the motors can be modeled as a matrix multiplication on the movement commands we issue, and we know that this can be corrected by another matrix multiplication. In order to support the capability to do this correction in hardware, in this part of the problem we want to design a circuit that implements a matrix multiplication.

Given two input voltages V_x and V_y (each with a source resistance of $2\text{k}\Omega$), design a circuit that produces $[V_{drvx}\ V_{drvy}]^T = A*[V_x\ V_y]$. You also **do not need to specify the values of the components nor implement any specific *A* matrix** – just make sure you label the components that would need to have different values. For the sake of simplicity, you can assume that all of the coefficients of the *A* matrix are positive, but otherwise can take on any value. **Hint:** what are the basic operations that need to be done to perform a matrix multiplication? If you're not sure where to start, draw these operations in block diagram form.

Matrix multiplication is just this:





PROBLEM 3. EECS Enrollments (19 points)

UC Berkeley's EECS department has seen incredible growth over the last few years, and as undergraduates all of you have unfortunately had to bear the brunt of the impacted classes. This problem explores the growth trends to some degree.

a) **(6 pts)** The following data is approximated from the Computing Research Association (CRA) website. The CRA surveyed departments across the US and looked at the average size of CS departments over many years. The observed the following average enrollment 2007 – 200, 2008 – 232, 2009 – 251, 2010 – 274, 2011 – 293, 2012 – 310. When you plot this data, you notice that the trend looks approximately linear with year. Given this observation, set up the measurement of department size vs. year as a matrix equation, and show how to get the coefficients of the best-fit line that minimizes the squared error between all the data points and the linear model that you could then use to predict the enrollment in 2013. (You don't need to provide any numerical answers – just set up the steps/equations.)

Model: enrollment = a $\cdot$ year + b

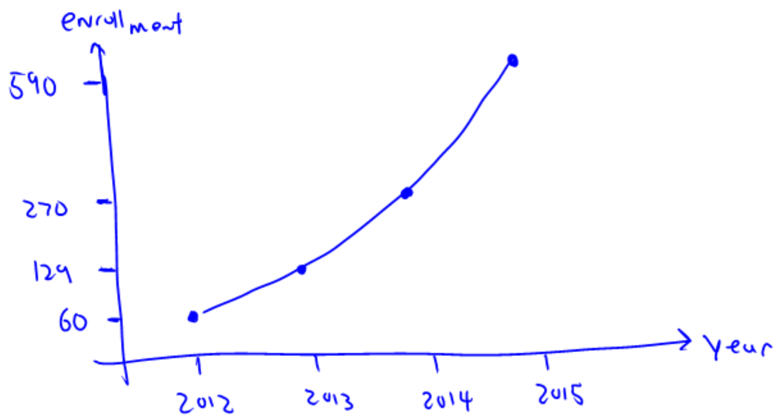
So, to find a and b:

$$\underbrace{\begin{bmatrix} 200 \\ 232 \\ 251 \\ 274 \\ 293 \\ 310 \end{bmatrix}}_{\vec{enr}} \approx \underbrace{\begin{bmatrix} 2007 & 1 \\ 2008 & 1 \\ 2009 & 1 \\ 2010 & 1 \\ 2011 & 1 \\ 2012 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}}$$

Least squares: $\begin{bmatrix} a_{ls} \\ b_{ls} \end{bmatrix} = (A^T A)^{-1} A^T \vec{enr}$

To predict 2013: enrollment $= a_{ls} \cdot 2013 + b_{ls}$

b) **(3 pts)** Since 2012 the growth has been even steeper. In fact, given the surging enrollments, a certain university decided to create a brand new freshman class, called EECS 116, to cater to the students. The first offering of EECS 116 (in 2012) had 60 students, the second (in 2013) had 129, the third (in 2014) had 270, and the fourth (in 2015) had 590 students enrolled. Plot this data. You should see an exponential growth in the enrollments – i.e. the data is of the form $a*2^{(b*x-c)}$, where a, b, and c are some constants, and x is related to the year.



c) **(10 pts)** Now let's look at using this data to predict the enrollment in 2016. In order to do this, you should first normalize the data by the enrollment in the first class. Then, **you should transform the measured data (and the model) using some function so that the result now depends linearly on the year and the parameters b and c**. Please clearly explain what function you would use to do this transformation, and then explain how you would use this transformed model to predict the enrollment for 2016.

* First, normalize so that enrollment data becomes

$$\begin{bmatrix} 1 & \frac{129}{60} & \frac{270}{60} & \frac{590}{60} \end{bmatrix}^T$$

* This normalization makes a fall out, so we only need to find b and c: $\frac{\text{enrollment}}{\text{enrollment}_{2012}} \propto 2^{bx-c}$

* To make the data depend linearly on b and c, we should take $\underline{\log_2()}$:

$$\underbrace{\begin{bmatrix} \log_2(1) \\ \log_2(\frac{129}{60}) \\ \log_2(\frac{270}{60}) \\ \log_2(\frac{590}{60}) \end{bmatrix}}_{\vec{enr}} = \underbrace{\begin{bmatrix} 2012 & -1 \\ 2013 & -1 \\ 2014 & -1 \\ 2015 & -1 \end{bmatrix}}_{A} \begin{bmatrix} b \\ c \end{bmatrix}$$

Use least squares again:

$$\begin{bmatrix} b_{ls} \\ c_{ls} \end{bmatrix} = (A^T A)^{-1} A^T \vec{enr}$$

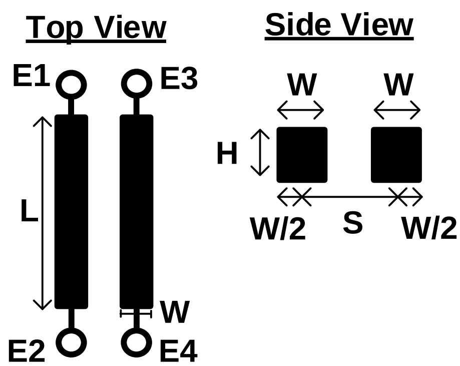
To predict 2016:

$$\text{enrollment} = 60 \cdot 2^{b_{ls} \cdot 2016 - c_{ls}}$$

PROBLEM 4. Temperature Sensing (20 pts)

In many of our devices we would like to have a means of measuring temperature electrically. For example, it is often very desirable to have devices shut themselves off when they get too hot, or a robot (like the ones we looked at in problems 1 and 2) may look for variations in environmental temperature to help it find interesting objects/areas to explore further.

Therefore, in this problem we'll look at how to realize some very simple circuits to produce a voltage that varies with temperature. The sensor will be based on utilizing the temperature dependence of the resistance or capacitance of the two rectangular pieces of metal arranged as shown below.



a) **(3 pts)** If all of the dimensions of the metal pieces (i.e., W, L, and H) depend on absolute temperature T as $X(T) = X_0 * (1 + k_d*T)$ (where k_d has units of $(°K)^{-1}$, and X can be W, L, or H) and the resistivity of the metal is $\rho(T) = \rho_0 * (1 + k_r*T)$ (where k_r also has units of $(°K)^{-1}$), write an expression for the resistance between E1 and E2 as a function of W_0, L_0, H_0 (i.e., the width, length, and height when T = 0°K), ρ_0, k_d, k_r, and T.

$$R = \rho \cdot \frac{L}{A} = \rho(T) \cdot \frac{L(T)}{W(T) \cdot H(T)}$$

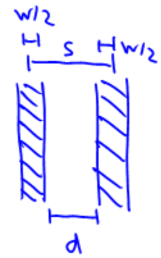
$$= \rho_0 (1 + k_r T) \cdot \frac{L_0 (1 + k_d T)}{W_0 (1 + k_d T) H_0 (1 + k_d T)}$$

$$\boxed{R(T) = \rho_0 \frac{L_0}{W_0 H_0} \cdot \frac{(1 + k_r T)}{(1 + k_d T)}} \quad \text{or} \quad R(T) = R_0 \cdot \frac{(1 + k_r T)}{(1 + k_d T)}$$

b) **(4 pts)** Assuming that the dimensions of the metals change as in part (a), but that the center-to-center spacing S and the permittivity ε do not change with temperature (you can assume that there is air in between the two pieces of metal), write an expression for the capacitance from E1 to E3 as a function of W0, L0, H0, S, ε, k_d, and T.

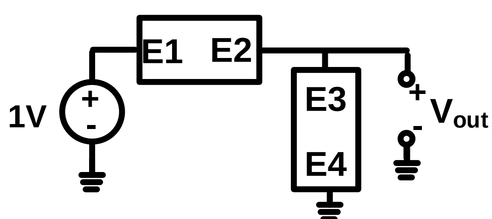
$$C = \varepsilon \cdot \frac{A}{d} = \varepsilon \cdot \frac{L(T) \cdot H(T)}{S - W(T)}$$

$$\boxed{C(T) = \varepsilon \cdot \frac{L_0(1+k_d T) H_0(1+k_d T)}{S - W_0(1+k_d T)}}$$



$d = S - W$

c) **(5 pts)** In order to convert the temperature-dependent resistance of the pieces of metal in to a temperature-dependent voltage, one of your colleagues (who hasn't taken EE16A) suggests that you connect the two pieces of metal in to a resistive divider circuit as shown below. However, your colleague builds this circuit and finds that V_{out} is **always** 0.5V – i.e., V_{out} does not depend on temperature at all! Using the same parameters/setup as part (a), derive an expression for V_{out} that shows why V_{out} does not depend on temperature.



$$R_{E1-E2}(T) = R_0 \cdot \frac{(1+k_r T)}{(1+k_d T)}$$

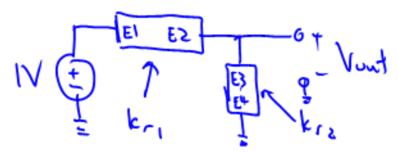
$$R_{E3-E4}(T) = R_0 \cdot \frac{(1+k_r T)}{(1+k_d T)}$$

$$V_{out} = \frac{R_{E3-E4}(T) \cdot 1V}{R_{E1-E2}(T) + R_{E3-E4}(T)} = \frac{R_0 \cdot \frac{(1+k_r T)}{(1+k_d T)} \cdot 1V}{2 R_0 \cdot \frac{(1+k_r T)}{(1+k_d T)}}$$

All the R_0's, $(1+k_r T)$, and $(1+k_d T)$ terms cancel: $V_{out} = \frac{1V}{2} = 0.5V$

d) **(8 pts)** If you had another type of metal available that has all of the same characteristics as the metal we used so far, except that it has a different k_r – i.e., for this metal, $\rho(T) = \rho0*(1+k_{r2}*T)$, propose a simple modification to the sensor design using both types of metals that would allow the circuit from (c) to produce a voltage that varies with temperature. Explain what the modification you would make is and provide an expression for V_{out} as a function of T.

Make one of the resistors out of metal #1, and the other out of metal #2:



$$R_{E1-E2} = R_0 \cdot \frac{(1+k_{r1}T)}{(1+k_l T)}$$

$$R_{E3-E4} = R_0 \cdot \frac{(1+k_{r2}T)}{(1+k_l T)}$$

$$V_{out} = \frac{R_{E3-E4} \cdot 1V}{R_{E1-E2} + R_{E3-E4}} = \frac{1+k_{r1}T}{1+k_{r1}T + 1+k_{r2}T} \cdot 1V$$

$$\boxed{V_{out}(T) = \frac{1+k_{r1}T}{2+(k_{r1}+k_{r2})T} \cdot 1V}$$

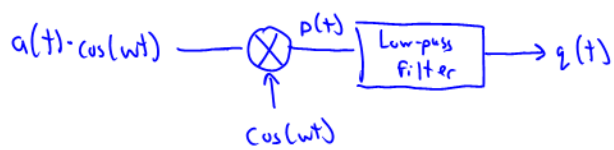
PROBLEM 5. AM Communication Systems (21 pts)

In this problem we'll look at a couple of different versions of AM communication.

To receive credit on this problem, you will need to provide brief explanations (either in words or using equations) to demonstrate that you understand how your block diagrams are intended to work.

Also, if any of your block diagrams includes a low-pass filter/moving average, you **do not** need to provide any details (such as cut-off frequency or averaging period) about them. Simply provide a clear explanation as to what you want to achieve by including such a filter.

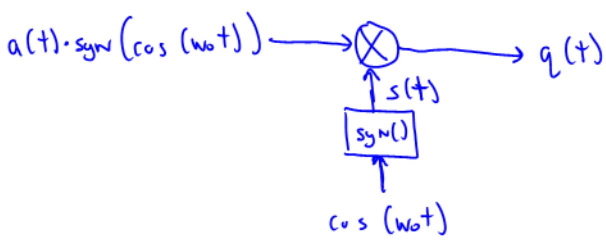
a) **(5 pts)** Let's first look at receiving an audio signal broadcast by an AM radio station. The station sends $x(t) = a(t) * \cos(\omega_0 * t)$, where $a(t) > 0$ and the highest non-zero magnitude frequency component associated with $a(t)$ (i.e., the bandwidth of $a(t)$) is at a dramatically lower frequency than ω_0. Assuming you have access to an oscillator that can produce exactly $\cos(\omega_0 * t)$ along with any other functions you would like, draw a block diagram of the receiver you would use to reproduce $a(t)$ at the output of the receiver.



$$p(t) = a(t)\cdot\cos(wt)\cdot\cos(wt) = a(t)\cdot\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2wt)\right)$$

If the low pass filter removes the cos(2wt) component and multiplies the low frequency part by 2, then $q(t) = a(t)$

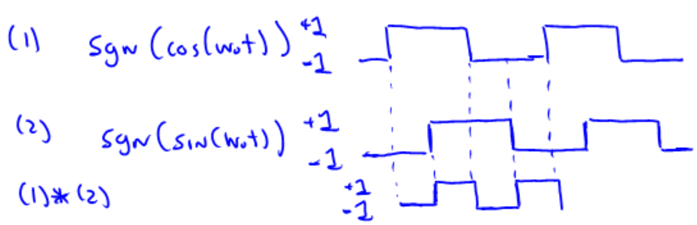
b) **(8 pts)** It turns out that in practice, generating sinusoidal waveforms efficiently is much harder than generating square waves. So, let's consider a new AM communication system where the radio station sends *x(t)* = *a(t)**sgn(cos(ω_0*t)), where the sgn(x) function is equal to 1 for x≥0 and -1 for x<0 (i.e., *a(t)* is multiplied by a square wave with angular frequency ω_0). Under the same conditions as in part (a) and assuming you have a block that implements the sgn() function, draw a block diagram of the receiver you would use to reproduce *a(t)*.
(**Hint**: Build on what you have done in class for cosine waves - the same principles apply here.)



A square wave [i.e., sgn(cos(w0t))] only takes on values of +1 and -1, so if we multiply the received signal by a square wave that is exactly aligned with the modulating square wave at the transmitter, the whole system multiplies a(t) by either 1·1 or (-1)·(-1) - i.e., in both cases we just multiply a(t) by 1! This is why we don't even need a low-pass filter in this case - a(t) shows up directly after the multiplication.

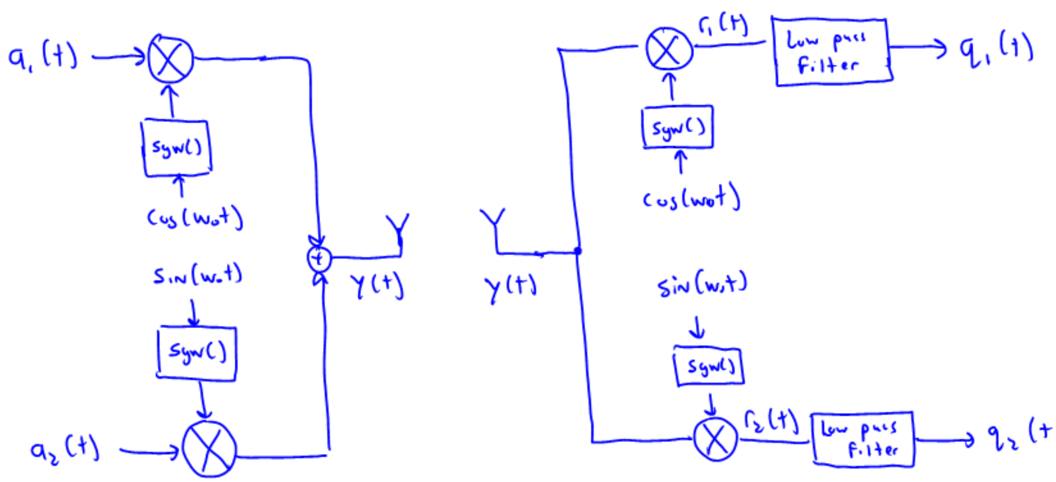
c) **(8 pts)** One of your colleagues claims that while still using square waves for the modulation, they can build a system that will transmit and receive two separate signals (e.g. to transmit the two channels of stereo audio) using only a single frequency oscillator and without the two signals interfering with each other. Assuming that both the transmitter and the receiver have access to oscillators that produce $\cos(\omega_0 * t)$ and/or $\sin(\omega_0 * t)$ (i.e., all oscillators produce exactly the same frequency) and also that you have as many copies of the sgn() block (as well as any other functions – e.g., add, multiply, etc.) as you would like, sketch a diagram of a communication system that can realize this goal. Be sure to show how your system ensures that the two separate signals do not interfere with each other.

The key is to realize that just like $\sin(\omega t)$ and $\cos(\omega t)$ are orthogonal to each other, $\mathrm{sgn}(\sin(\omega t))$ and $\mathrm{sgn}(\cos(\omega t))$ are also orthogonal. To see this, let's plot each of them & their product:



the average value of this over one period is 0!

So, the system looks like this:



* Look at e.g. $r_1(t) = \{a_1(t)\,\mathrm{sgn}[\cos(\omega_0 t)] + a_2(t)\,\mathrm{sgn}[\sin(\omega_0 t)]\} \cdot \mathrm{sgn}[\cos(\omega_0 t)]$

$= a_1(t)\,\underbrace{\mathrm{sgn}[\cos]\cdot\mathrm{sgn}(\cos]}_{\text{always } = 1} + a_2(t)\,\underbrace{\mathrm{sgn}[\sin]\,\mathrm{sgn}[\cos]}_{\text{removed by low-pass filter}}$

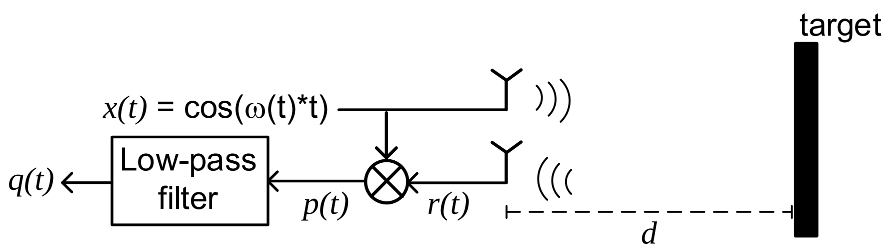
↳ So: $q_1(t) = a_1(t)$

↳ Note that same argument shows that $q_2(t) = a_2(t)$; just need to realize that $\mathrm{sgn}[\cos]\,\mathrm{sgn}[\sin] = 1$

PROBLEM 6. Radar (10 pts + BONUS 14 pts)

The modulation and demodulation techniques we used for wireless communication can also be used to build radar systems, which sense the distance between a "target" object and the sensor.

A simplified block diagram of the basic setup for such a radar is shown below. The radar produces a (relatively) high frequency sinusoid $x(t) = \cos(\omega(t)*t)$ (where $\omega(t)$ is a frequency that could be varying with time) as an electromagnetic wave that propagates through space for some distance d until it hits a stationary target. At this point, the wave reflects off of the target, and travels distance d back to the the radar where it is received as $r(t)$. In this problem, we'll assume that $r(t) = x(t-2*t_d)$, where t_d is the time it takes for the wave to travel the distance d from the radar to the target.



As we will see in further detail shortly, to extract information about the distance to the target, $r(t)$ is multiplied by the originally transmitted signal $x(t)$ (resulting in $p(t)$) and then low-pass filtered to remove all of the high-frequency components of $p(t)$. Note that for this problem you can assume that the target is completely stationary relative to the radar – i.e., d does not vary with time.

a) **(6 pts)** If $\omega(t)$ is constant with time – in particular, $\omega(t) = \omega_0 = 2\pi*5\text{GHz}$ (i.e., the radar transmits a cosine wave with a constant frequency of 5GHz) – write an expression for $q(t)$ as a function of the distance to the target d. Note that the velocity of an electromagnetic wave is the same as the speed of light – i.e., 3e8 m/s, and that the low-pass filter completely eliminates any signals with frequencies near $2\omega_0$.

$r(t) = x(t-2t_d)$ $\qquad$ $t_d = \frac{d}{3e8}$

$= \cos(2\pi \cdot w_0(t-2t_d))$

this is constant with time

$p(t) = \cos(2\pi w_0(t-2t_d)) \cdot \cos(2\pi w_0 t) = \frac{1}{2}\cos(4\pi w_0 t_d) + \frac{1}{2}\cos\big(\underbrace{2\pi w_0(2t-2t_d)}_{\text{angular frequency is } 2w_0}\big)$

Low-pass filter removes $2w_0$ component:

$q(t) = \frac{1}{2}\cos(4\pi w_0 t_d) \longrightarrow q(t) = \frac{1}{2}\cos\left(4\pi \cdot 5e9 \cdot \frac{d}{3e8}\right)$

$\boxed{q(t) = \frac{1}{2}\cos\left(\frac{200}{3}\pi \cdot d\right)}$

b) **(4 pts)** For the same setup as in part (a), if *q(t)* = 0.5*cos(π), what are **all of the distances** *d* that the target could have been at? (Hint: what is the period of cos(x)?)

Period of cos(x) is $\underline{2\pi}$, so $\cos(x+2\pi k)=\cos(x)$ for integer k

$$\frac{1}{2}\cos(\pi+2\pi k)=\frac{1}{2}\cos\left(\frac{200}{3}\pi\cdot d\right)$$

$$\pi+2\pi k=\frac{200}{3}\pi d$$

$$d=\frac{3}{200}+\frac{6}{200}k \text{ , k integer } \& \geq 0$$

$$\boxed{d=1.5\text{cm}+3\text{cm}\cdot k\text{, k integer } \& \geq 0}$$

c) **(BONUS: 8 pts)** In order to uniquely determine a single distance between the radar and the target, instead of using a constant frequency, we can instead introduce frequency modulation. In particular, assuming that ω(t) = $\omega_0 + \Delta\omega*(t/T_0)$, where T_0 is some arbitrary scale factor with units of seconds, and that $\Delta\omega*(t/T_0)<<2\omega_0$, write a new expression for *q(t)* as a function of distance *d*.

$$r(t)=\cos\left[w(t-2t_d)\cdot(t-2t_d)\right]$$

$$p(t)=r(t)\cdot\cos\left[w(t)\cdot t\right]=\frac{1}{2}\cos\left[w(t)\cdot t-w(t-2t_d)\cdot(t-2t_d)\right]+\underbrace{\frac{1}{2}\cos\left[w(t-2t_d)\cdot(t-2t_d)+w(t)\cdot t\right]}_{\text{removed by low-pass filter}}$$

$$w(t-2t_d)\cdot(t-2t_d)=w_0(t-2t_d)+\Delta w\frac{(t-2t_d)^2}{T_0} \quad (1)$$

$$w(t)\cdot t=w_0t+\Delta w\frac{t^2}{T_0} \quad (2)$$

$$(2)-(1)=2w_0t_d+\Delta w\frac{4t_dt}{T_0}-\Delta w\frac{4t_d^2}{T_0}$$

So, $q(t)=\frac{1}{2}\cos\left[4\Delta w\frac{t_d}{T_0}\cdot t+2w_0t_d-4\Delta w t_d\cdot\frac{t_d}{T_0}\right]$, where $t_d=\frac{d}{3e8}$

d) **(BONUS: 6 pts)** Given your answer to part c), to extract the distance *d*, what property of the periodic signal *q(t)* should you measure?

$q(t)$ can be rewritten as:

$$q(t) \propto \cos\left[\underbrace{4\Delta\omega\frac{t_d}{T_0}}\cdot t + \phi\right]$$

This sets the frequency of $q(t)$, and is directly proportional to d (since $t_d = \frac{d}{3e8}$).

where ϕ does not depend on time (even though it does depend on t_d, ω_0, $\Delta\omega$, and T_0)

So, all we need to do is measure the frequency of $q(t)$ and scale it appropriately to find d.

PROBLEM 7. Filtering Out Interference (24 pts)

In this problem we'll look at using filtering to make wireless systems (e.g., for AM communication) more robust to potential interference from other wireless systems operating at nearby frequencies.

Throughout this problem we'll be working in discrete time. You can assume that the highest frequency component of the signal we care about is at π/25 rad/sample, and that the interference from the other wireless system is a sinusoid at π/4 rad/sample.

a) **(4 pts)** Let's first consider a 3-point moving-average filter – i.e., $y(n) = 1/3*[x(n) + x(n-1) + x(n-2)]$ – as a potential candidate to reduce the amount of interference relative to the desired signal. Is this filter a linear time invariant system? You should either prove that the system is LTI, prove that it isn't, or show why you have insufficient information to conclude one way or the other.

Check linearity: Input $\hat{x}(n) = \alpha x_1(n) + \beta x_2(n)$

$$\hat{y}(n) = \frac{1}{3}\left[\alpha x_1(n) + \beta x_2(n) + \alpha x_1(n-1) + \beta x_2(n-1) + \alpha x_1(n-2) + \beta x_2(n-2)\right]$$

$$\hat{y}(n) = \frac{1}{3}\left[\alpha x_1(n) + \alpha x_1(n-1) + \alpha x_1(n-2)\right] + \frac{1}{3}\left[\beta x_2(n) + \beta x_2(n-1) + \beta x_2(n-2)\right]$$

$$= y(n)\Big|_{\alpha x_1(n)} + y(n)\Big|_{\beta x_2(n)}$$

The system is linear.

Check time-invariance: If $\hat{x}(n) = x(n-m)$, $\hat{y}(n) = \frac{1}{3}\left[x(n-m) + x(n-m-1) + x(n-m-2)\right]$

$$y(n-m) = \frac{1}{3}\left[x(n-m) + x(n-m-1) + x(n-m-2)\right] = \hat{y}(n)$$

The system is time-invariant.

b) **(5 points)** To help us predict how well this 3-point moving-average filter will eliminate the interference, determine the frequency response $H_{3pt}(\omega)$ for this filter.

$$H_{3pt}(w) = \frac{1}{3}\left[e^{j0} + e^{-jw} + e^{-2jw}\right]$$

$$= \boxed{\frac{1}{3}\left[1 + e^{-jw} + e^{-2jw}\right]}$$

c) **(3 points)** Determine the response of the same 3-point moving-average filter (i.e. calculate the output of the filter) to the interfering input $x_{interferer}(n) = A_{int} * e^{(j*\pi/4*n)}$.

$$W = \frac{\pi}{4}, \text{ so } \quad H\left(\frac{\pi}{4}\right) = \frac{1}{3}\left[1 + e^{-j\pi/4} + e^{-j2\pi/4}\right]$$

$$\boxed{y(n) = A_{int} \cdot \frac{1}{3}\left[1 + e^{-j\pi/4} + e^{-j2\pi/4}\right] \cdot e^{j(\pi/4)n}}$$

d) **(6 pts)** Now let's consider a 4-point moving-average filter $y(n)$ = 1/4*[$x(n)$ + $x(n\text{-}1)$ + $x(n\text{-}2)$ + $x(n\text{-}3)$]. What is the output of this filter for the same interfering input signal $x_{interferer}(n)$?

$$\boxed{y(n) = \frac{1}{4}\left[1 + e^{-j(\pi/4)} + e^{-j(2\pi/4)} + e^{-j(3\pi/4)}\right] \cdot A_{int}\, e^{j(\pi/4)n}}$$

e) **(6 pts)** If you wanted to make your system as immune to interference as possible and had to choose between the 3-point moving-average filter and the 4-point moving-average filter, which one would you select, and why?

Let's look at the magnitude of the two responses:

$$\frac{1}{3}\cdot\left[1 + e^{-j\pi/4} + e^{-j2\pi/4}\right] = \frac{1}{3}\left(1 + \cos(\pi/4) - j\sin(\pi/4) + \cos(2\pi/4) - j\sin(2\pi/4)\right)$$

$$= \frac{1}{3}\left[1 + \cos(\pi/4) + \cos(2\pi/4) - j\left(\sin(\pi/4) + \sin(2\pi/4)\right)\right]$$

$$= \frac{1}{3}\left[1 + \frac{1}{\sqrt{2}} - j\left(\frac{1}{\sqrt{2}} + 1\right)\right]$$

$$|H_{3pt}| = \frac{1}{3}\cdot\left(\sqrt{2} + 1\right) \approx 0.8$$

$$\frac{1}{4}\left[1 + e^{-j\pi/4} + e^{-j2\pi/4} + e^{-j3\pi/4}\right] = \frac{1}{4}\left[1 + \frac{1}{\sqrt{2}} \underbrace{- \frac{1}{\sqrt{2}}}_{\cos(3\pi/4)} - j\left(\frac{1}{\sqrt{2}} + 1 + \underbrace{\frac{1}{\sqrt{2}}}_{\sin(3\pi/4)}\right)\right]$$

$$= \frac{1}{4}\left[1 - j\left(1 + \sqrt{2}\right)\right]$$

$$|H_{4pt}| = \frac{1}{4}\sqrt{1 + (1+\sqrt{2})^2} \approx 0.65$$

So, choose the <u>4-point</u> filter.