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EECS 16A    Designing Information Devices and Systems I  
Spring 2019 Final Exam

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Final Exam Solution

PRINT your student ID: \_\_\_\_\_

PRINT AND SIGN your name: \_\_\_\_\_, \_\_\_\_\_  
(last name)                      (first name)                      (signature)

PRINT your discussion section and GSI(s) (the one you attend): \_\_\_\_\_

Name and SID of the person to your left: \_\_\_\_\_

Name and SID of the person to your right: \_\_\_\_\_

Name and SID of the person in front of you: \_\_\_\_\_

Name and SID of the person behind you: \_\_\_\_\_

**1. What was your favorite part of EE 16A? (1 point)**

**2. What are you looking forward to over the summer break? (1 point)**

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

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### 3. Sensor Calibration (20 points)

You are setting up a lab experiment that involves measuring the position  $s$  of an object on a line. For this task, you are given a sensor that outputs a voltage based on the position of an object, as shown in Figure 3.1. Your goal is to determine a function that returns the best approximation of the object's position based on the sensor's output voltage. You measure the output voltage of the sensor for some positions and get the data in Table 3.1.

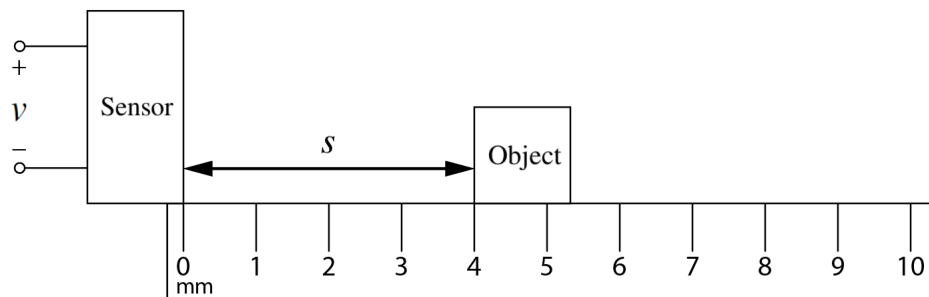


Figure 3.1: Experimental setup for sensor calibration.

Object's Position $s$ (mm)	1	2	3
Sensor's Output Voltage $v$ (mV)	4	10	15

Table 3.1: Sensor measurements for part (a).

Recall that  $m$  is *milli-*, or  $10^{-3}$ .

- (a) (4 points) After plotting the data from Table 3.1, you notice it is fairly linear. So, you first assume that position is approximated by scaled voltage:

$$s \approx kv$$

where  $k$  is a scalar constant. Notice that the points don't exactly lie on a line due to noise. **Find  $k$  that minimizes the squared error  $\|\vec{s} - k\vec{v}\|^2$ .**

**Solution:** We want to find the approximate solution to the following equation:

$$\begin{bmatrix} 4 \\ 10 \\ 15 \end{bmatrix} k = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Next, we solve for  $k$  using least squares:

$$\begin{aligned} \vec{x} &= (A^T A)^{-1} A^T \vec{b} \\ k &= \left( \begin{bmatrix} 4 & 10 & 15 \end{bmatrix} \begin{bmatrix} 4 \\ 10 \\ 15 \end{bmatrix} \right)^{-1} \begin{bmatrix} 4 & 10 & 15 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ k &= \frac{69}{341} \end{aligned}$$

- (b) (6 points) You measure the output voltage to be  $-3$  at  $s = 0$ , so you realize there must be a bias on the sensor. Your updated data is in Table 3.2.

Object's Position $s$ (mm)	0	1	2	3
Sensor's Output Voltage $v$ (mV)	-3	4	10	15

Table 3.2: Sensor measurements for part (b).

You revise your equation as follows:

$$s \approx kv + b$$

where  $b$  is a scalar constant representing the bias. **Set up the least squares problem to find  $k$  and  $b$  using data from Table 3.2. Derive an expression for  $k$  and  $b$  (i.e.,  $\begin{bmatrix} k \\ b \end{bmatrix} = ???$ ), as a function of table data, but DO NOT SOLVE.**

**Solution:** We want to find the approximate solution to the following equation:

$$\begin{bmatrix} -3 & 1 \\ 4 & 1 \\ 10 & 1 \\ 15 & 1 \end{bmatrix} \begin{bmatrix} k \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

We express  $k$  and  $b$  using least squares:

$$\vec{x} = (A^T A)^{-1} A^T \vec{b}$$

$$\begin{bmatrix} k \\ b \end{bmatrix} = \left( \begin{bmatrix} -3 & 4 & 10 & 15 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 4 & 1 \\ 10 & 1 \\ 15 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -3 & 4 & 10 & 15 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

- (c) (6 points) Using a different sensor, you take more data points and find that these points don't follow your model from (b) very well:

Object's Position $s$ (mm)	4	5	6	7
Sensor's Output Voltage $v$ (mV)	20	45	98	214

Table 3.3: Sensor measurements for part (c).

By inspection, you decide either a quadratic fit or a logarithmic fit may work. The equations for these mathematical models are as follows:

$$s_{quadratic} \approx a_2 v^2 + a_1 v + a_0$$

$$s_{logarithmic} \approx b_1 \ln(v) + b_0$$

where  $a_0$ ,  $a_1$ ,  $a_2$ ,  $b_0$ , and  $b_1$  are constants. **Set up the least squares problems for the two models using data only from Table 3.3. Derive expressions for the constants (i.e.,  $\begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = ???$  and  $\begin{bmatrix} b_1 \\ b_0 \end{bmatrix} = ???$ ), as functions of table data, but DO NOT SOLVE.**

**Solution:** For the quadratic fit, we want to find the approximate solution to the following equation:

$$\begin{bmatrix} 20^2 & 20 & 1 \\ 45^2 & 45 & 1 \\ 98^2 & 98 & 1 \\ 214^2 & 214 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$$

We approximately solve for the constants using least squares:

$$\begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = (A^T A)^{-1} A^T \vec{b}$$

For the logarithmic fit, we want to find the approximate solution to the following equation:

$$\begin{bmatrix} \ln(20) & 1 \\ \ln(45) & 1 \\ \ln(98) & 1 \\ \ln(214) & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$$

We approximately solve for the constants using least squares:

$$\begin{bmatrix} b_1 \\ b_0 \end{bmatrix} = (A^T A)^{-1} A^T \vec{b}$$

- (d) (4 points) After fitting the models from part (c), assume you calculate the following errors  $e = s_{\text{approximate}} - s_{\text{measured}}$  for the sets of data in that part:

Object's Position $s$ (mm)	4	5	6	7
$e_{\text{quadratic}}$ (mm)	1	-1.5	0.5	0
$e_{\text{logarithmic}}$ (mm)	-0.9	0	-1	1.1

**In the context of least squares, which is the better fit? Justify your answer mathematically.**

**Solution:** In least squares, we aim to minimize the cost function

$$\text{Cost} = \left\| A\vec{x} - \vec{b} \right\|^2$$

We calculate the cost for each model:

$$\text{Cost}_{\text{quadratic}} = 1^2 + (-1.5)^2 + 0.5^2 + 0^2 = 3.5$$

$$\text{Cost}_{\text{logarithmic}} = (-0.9)^2 + 0^2 + (-1)^2 + 1.1^2 = 3.02$$

Since the cost of the logarithmic fit is smaller than that of the quadratic fit, we conclude that the logarithmic fit is better.

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#### 4. I think *neuron* to something! (18 points)

Figure 4.1 shows a diagram of a neural cell membrane. There are pumps and channels for both sodium ( $\text{Na}^+$ ) and potassium ( $\text{K}^+$ ).

Despite their names, the channels act as current sources while the pumps behave like resistors. The membrane itself acts as a capacitor, as it is able to build up charge on either side.

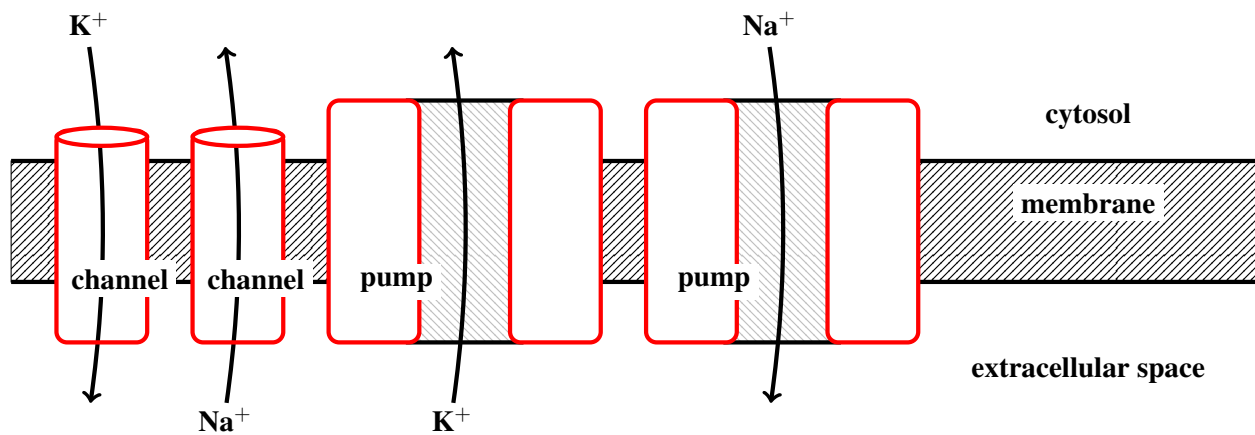


Figure 4.1: Diagram of the neural membrane

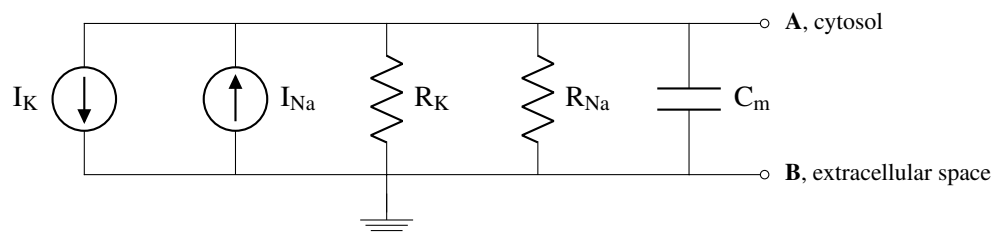


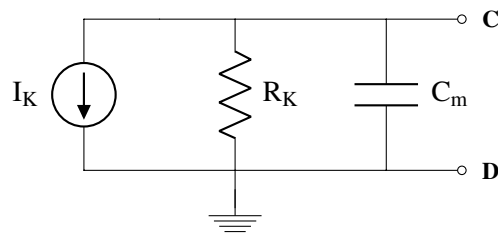
Figure 4.2: Circuit model of the neural membrane

- (a) (3 points) In Figure 4.2, explain why, if we attach different loads and measure the currents and voltages through the loads, we will not be able to find values for  $I_K$  and  $R_K$ .

**Solution:**

We would only be able to measure some  $I_K = I_{\text{Norton}} + I_{\text{Na}}$ , with  $I_{\text{Na}}$  acting as a free variable since  $I_{\text{Norton}} = I_{\text{Na}} - I_K$ . Similar reasoning for  $R_K \parallel R_{\text{Na}}$ .

- (b) (4 points) Thankfully, some of your biologist friends know how to remove the sodium channel and pump so you can measure  $I_K$  and  $R_K$ . Doing this, you are left with the following:

Figure 4.3: Membrane with sodium ( $\text{Na}^+$ ) complexes removed

At steady state, you measure an open-circuit voltage of  $V_{\text{CD}} = -100 \text{ mV}$ . Also at steady state, you find a short-circuit current of  $I_{\text{CD}} = -50 \text{ pA}$ . **Find  $I_{\text{K}}$  and  $R_{\text{K}}$ .**

*Hint: recall that  $p$  (pico) is  $10^{-12}$ .*

**Solution:**

When short-circuited,  $I_{\text{K}} = -I_{\text{sc}} = 50 \text{ pA}$ .

When open-circuited at steady state, the capacitor is fully charged with no current.

$$\begin{aligned} R_{\text{K}} &= R_{\text{th}} = V_{\text{oc}} / I_{\text{sc}} = 100 \text{ mV} / 50 \text{ pA} \\ &= 2 \cdot 10^9 \Omega = 2 \text{ G}\Omega. \end{aligned}$$

- (c) (4 points) Similarly, you find that  $I_{\text{Na}} = \frac{1}{100} \cdot I_{\text{K}}$  and  $R_{\text{Na}} = 100 \cdot R_{\text{K}}$ . **Find  $V_{\text{AB}}$  in Figure 4.2 in terms of  $I_{\text{K}}$  and  $R_{\text{K}}$ .**

**Solution:**

$$\begin{aligned} V_{\text{AB}} &= (I_{\text{Na}} - I_{\text{K}}) \cdot R_{\text{Na}} \parallel R_{\text{K}} \\ &= (I_{\text{Na}} - I_{\text{K}}) \cdot \frac{R_{\text{Na}} \cdot R_{\text{K}}}{R_{\text{Na}} + R_{\text{K}}} \\ &= -\frac{99}{100} \cdot I_{\text{K}} \cdot \frac{100 \cdot R_{\text{K}}^2}{101 \cdot R_{\text{K}}} \\ &= -\frac{99}{101} \cdot I_{\text{K}} \cdot R_{\text{K}}. \end{aligned}$$

- (d) (7 points) Due to the high values of the resistors and the solution to part (c), we will now neglect the resistors and passive sodium channels.

Neurons transmit signals from one to the other via neurotransmitters. When a nearby neuron releases the neurotransmitter acetylcholine, special acetylcholine-gated sodium channels open wide, allowing a large amount of sodium  $I_{\text{Na}}^{(\text{d})}$  to rush into the cytosol.

If  $I_{\text{Na}}^{(\text{d})}$  is on for long enough, the membrane voltage changes enough that it reaches a critical value  $V_{\text{ref}}$ , triggering a neural response.

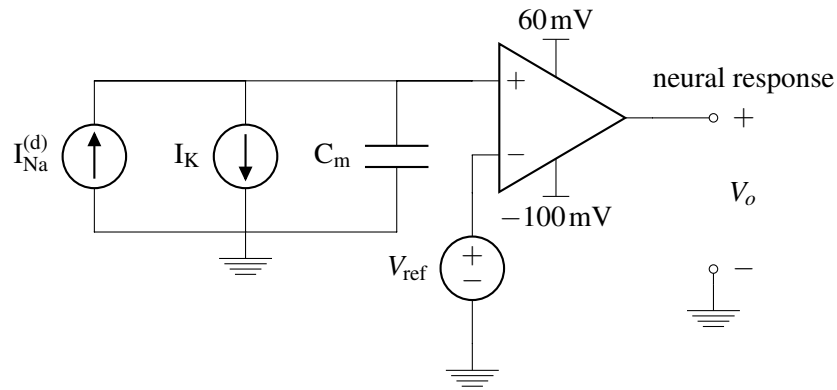


Figure 4.4: Triggerable membrane

Assuming:

- $I_K = 50 \text{ pA}$
- $I_{\text{Na}}^{(d)} = 150 \text{ pA}$
- $C_m = 20 \text{ pF}$
- $V_{\text{ref}} = -50 \text{ mV}$
- $V_+$  is initially  $-100 \text{ mV}$ ,

**How long does it take to trigger a response from the neuron?**

**Solution:**

This is essentially the capacitive touchscreen lab.

There is a current of  $I_{\text{Na}}^{(d)} - I_K$  flowing down the capacitor, so the voltage at the positive terminal has

$$\frac{dV^+}{dt} = \frac{1}{C_m} (I_{\text{Na}}^{(d)} - I_K).$$

For a constant current, this becomes

$$\frac{\Delta V^+}{\Delta t} = \frac{1}{C_m} (I_{\text{Na}}^{(d)} - I_K) \implies \Delta t = C_m \cdot \Delta V^+ \cdot \frac{1}{I_{\text{Na}}^{(d)} - I_K},$$

where  $\Delta V^+ = V_{\text{ref}} - V_0^+$ .

$$\Delta t = 20 \text{ pF} \cdot (-50 \text{ mV} + 100 \text{ mV}) \cdot \frac{1}{150 \text{ pA} - 50 \text{ pA}} = 10 \text{ ms}.$$

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### 5. Tracking your TA Terry (10 points)

Terry is a very mischievous child, and his mother is interested in tracking him. Terry sends meetup locations in the form of a **two-dimensional real vector** to his friends. His mother is convinced he is encoding his messages in a different basis, but she doesn't quite know how, so she asks for your help in figuring it out.

For this problem, the  $\mathbb{R}^2$  standard basis vectors will be denoted by

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- (a) (2 points) You are given that the text sent by Terry is  $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ . The mom only knows the **standard basis**, while Terry is using basis vectors  $\vec{v}_1$  and  $\vec{v}_2$ . Based on this information, **write the location each person thinks the text refers to as a linear combination of the given basis vectors:  $\vec{e}_1, \vec{e}_2, \vec{v}_1, \vec{v}_2$ . Label your answer for the mother as  $\vec{r}_m$  and  $\vec{r}_t$  for Terry.**

**Solution:** The location in his mom's perspective is simply the location that was given in the text, because she uses the standard basis:  $\vec{r}_m = 2\vec{e}_1 + 3\vec{e}_2$ .

The location in Terry's perspective is that of  $\vec{r}_t = 2\vec{v}_1 + 3\vec{v}_2$ .

- (b) (3 points) Now you're told that Terry sent a text  $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ . Terry's friend tells you that Terry's location is given by the vector  $\begin{bmatrix} 8 \\ 9 \end{bmatrix}$ . Assuming this vector represents the standard basis coordinates of Terry's actual location, **determine the unique basis vectors he is using, or if it is impossible—explain why.**

**Solution:** Solving for the basis vectors Terry is using for this transformation (or in other words how he sees things in his coordinate space) is akin to determining the values of  $x_1, x_2, x_3, x_4$  in the matrix vector equation:

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

There are clearly 4 unknowns and only two equations, so this task is impossible.

- (c) (5 points) As it turns out, Terry is a nice guy and is not doing anything mischievous after all. To prove it, he decides to help his mom learn how he's encoding his messages. He gives his mom a new set of basis vectors **P**:

$$\vec{p}_1 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \text{ and } \vec{p}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

and tells her his own basis vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

He challenges her to determine the coordinates (in the **P** basis), corresponding to the location  $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$  in his own basis. **Determine the coordinates, or if the task is impossible—explain why.**



**Solution:** The problem can be formulated as a change of basis problem where the location in reality is the equality:

$$\begin{bmatrix} 0 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Solving this results in

$$\begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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### 6. Automatic Bitcoin Investment Portfolio (18 points)

Your GSI Sang Min hears about the hype that surrounds Bitcoin and decides to invest some of his salary in Bitcoin. Sang Min is extremely lazy and wants to choose from one of two automatic payment systems that buy a set amount of money's worth of Bitcoin at the beginning of every quarter in the financial calendar year. The systems then sell the Bitcoin and return to Sang Min the money earned from selling that amount of Bitcoin at the end of every quarter. The following are the two available investment portfolio choices:

$$\text{Portfolio A : } \vec{p}_A = [\$60 \quad \$40 \quad \$60 \quad \$40]^T$$

$$\text{Portfolio B : } \vec{p}_B = [\$20 \quad \$80 \quad \$20 \quad \$80]^T.$$

If Sang Min selects Portfolio A, \$60 of Sang Min's money is invested in Bitcoin at the start of the first quarter, \$40 at the start of the second quarter, \$60 at the start of the third quarter, and \$40 at the start of the fourth quarter. Similarly, selecting Portfolio B invests \$20 in the first quarter, \$80 in the second quarter, and repeats.

The two portfolios above also offer Sang Min the choice of investing the first amount at the start of the second quarter, i.e. Sang Min can choose to invest \$60 in the first quarter, \$40 in the second quarter, and repeat **or** invest \$40 in the first quarter, \$60 in the second quarter, and repeat, if Portfolio A is selected.

The projected Bitcoin valuation at the end of a quarter with respect to its valuation at the beginning of the quarter for the next year in quarterly increment is represented by  $\vec{x}$  below:

$$\vec{x} = [+0.5 \quad -1 \quad +0.5 \quad -2]^T.$$

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- (a) (6 points) Recall that  $\vec{x} = [+0.5 \ -1 \ +0.5 \ -2]^T$ . **Compute the circular cross-correlation of  $\vec{x}$  and  $\vec{p}_A$  with period  $N = 4$ , i.e. find  $\text{circcorr}(\vec{x}, \vec{p}_A)[k]$ , for  $k = 0, 1, 2, 3$ . Similarly, compute  $\text{circcorr}(\vec{x}, \vec{p}_B)[k]$  for  $k = 0, 1, 2, 3$ .**

**Solution:** The circular cross-correlation of  $\vec{x}$  and  $\vec{p}_A$  is defined as follows:

$$\text{circcorr}(\vec{x}, \vec{p}_A)[k] = \sum_{i=0}^{N-1} x[i]p_A[(i-k)_N].$$

Therefore, we have

$\vec{x}$	0.5	-1	0.5	-2	
$\vec{p}_A[(i-0)_4]$	\$60	\$40	\$60	\$40	
$\langle \vec{x}, \vec{p}_A[(i-0)_4] \rangle$	\$30	- \$40	+ \$30	- \$80	= -\$60

$\vec{x}$	0.5	-1	0.5	-2	
$\vec{p}_A[(i-1)_4]$	\$40	\$60	\$40	\$60	
$\langle \vec{x}, \vec{p}_A[(i-1)_4] \rangle$	\$20	- \$60	+ \$20	- \$120	= -\$140

$\vec{x}$	0.5	-1	0.5	-2	
$\vec{p}_A[(i-2)_4]$	\$60	\$40	\$60	\$40	
$\langle \vec{x}, \vec{p}_A[(i-2)_4] \rangle$	\$30	- \$40	+ \$30	- \$80	= -\$60

$\vec{x}$	0.5	-1	0.5	-2	
$\vec{p}_A[(i-3)_4]$	\$40	\$60	\$40	\$60	
$\langle \vec{x}, \vec{p}_A[(i-3)_4] \rangle$	\$20	- \$60	+ \$20	- \$120	= -\$140

Similarly, the circular cross-correlation of  $\vec{x}$  and  $\vec{p}_B$  is defined as follows:

$$\text{circcorr}(\vec{x}, \vec{p}_B)[k] = \sum_{i=0}^{N-1} x[i]p_B[(i-k)_N].$$

Therefore, we have

$\vec{x}$	0.5	-1	0.5	-2	
$\vec{p}_B[(i-0)_4]$	\$20	\$80	\$20	\$80	
$\langle \vec{x}, \vec{p}_B[(i-0)_4] \rangle$	\$10	- \$80	+ \$10	- \$160	= -\$220

$\vec{x}$	0.5	-1	0.5	-2	
$\vec{p}_B[(i-1)_4]$	\$80	\$20	\$80	\$20	
$\langle \vec{x}, \vec{p}_B[(i-1)_4] \rangle$	\$40	- \$20	+ \$40	- \$40	= \$20

$\vec{x}$	0.5	-1	0.5	-2	
$\vec{p}_B[(i-2)_4]$	\$20	\$80	\$20	\$80	
$\langle \vec{x}, \vec{p}_B[(i-2)_4] \rangle$	\$10	- \$80	+ \$10	- \$160	= -\$220

$\vec{x}$	0.5	-1	0.5	-2	
$\vec{p}_B[(i-3)_4]$	\$80	\$20	\$80	\$20	
$\langle \vec{x}, \vec{p}_B[(i-3)_4] \rangle$	\$40	- \$20	+ \$40	- \$40	= \$20

- (b) (2 points) If Sang Min chooses Portfolio A, **which option (\$60 first or \$40 first)** should he take to maximize profit/minimize loss after one year? *Please bubble in your answer completely below and explain your answer.*

\$60 first

\$40 first

If Sang Min chooses Portfolio B, **which option (\$20 first or \$80 first)** should he take to maximize profit/minimize loss after a year? *Please bubble in your answer completely below and explain your answer.*

\$20 first

\$80 first

**Solution:** For Portfolio A, Sang Min should take the option of investing \$60 first because his return is -\$60 after a year as opposed to the much worse return of -\$140. For Portfolio B, Sang Min should take the option of investing \$80 first because he receives a return of \$20 after the one year period as opposed to having a return of -\$220.

- (c) (2 points) **Which portfolio and option** should Sang Min choose to maximize profit/minimize loss after a year? **How much money** did Sang Min make/lose over the year with this optimal choice? [*Hint: Sang Min invested \$200 over the one year period.*]

**Solution:** Sang Min should invest in Portfolio B and take the option of investing \$80 first. With this choice and option, Sang Min minimizes his loss, which equals  $\$200 - \$20 = \$180$ . Another acceptable interpretation of  $\vec{x}$  is the following: the elements of  $\vec{x}$  can be interpreted as the increment from the start of a quarter to the end of a quarter. Then,  $\langle \vec{x}, \vec{p}_B[(i-1)_4] \rangle$  represents the “profit” on top of the investment amount of \$200. With this interpretation, Sang Min gains a profit of \$20 on top of the \$200 invested with this choice and option.

*For the remaining parts of this problem, consider the scenario where Sang Min chooses Portfolio A.*

- (d) (4 points) Write the circular cross-correlation vector of  $\vec{x}$  and  $\vec{p}_A$  as a matrix-vector product, i.e. write down the matrix  $\mathbf{C}$  in the equation

$$\overrightarrow{\text{circcorr}}(\vec{x}, \vec{p}_A) = \mathbf{C}\vec{x}.$$

**Solution:** We have

$$\begin{bmatrix} \langle \vec{x}, \vec{p}_A[(i-0)_4] \rangle \\ \langle \vec{x}, \vec{p}_A[(i-1)_4] \rangle \\ \langle \vec{x}, \vec{p}_A[(i-2)_4] \rangle \\ \langle \vec{x}, \vec{p}_A[(i-3)_4] \rangle \end{bmatrix} = \mathbf{C} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}.$$

Therefore,

$$\mathbf{C} = \begin{bmatrix} p_A[0] & p_A[1] & p_A[2] & p_A[3] \\ p_A[3] & p_A[0] & p_A[1] & p_A[2] \\ p_A[2] & p_A[3] & p_A[0] & p_A[1] \\ p_A[1] & p_A[2] & p_A[3] & p_A[0] \end{bmatrix} = \begin{bmatrix} 60 & 40 & 60 & 40 \\ 40 & 60 & 40 & 60 \\ 60 & 40 & 60 & 40 \\ 40 & 60 & 40 & 60 \end{bmatrix}.$$

During one meeting with Professor Liu, Sang Min learns that Professor Liu made thousands of dollars in 2016 using Portfolio A. Sang Min is curious about the quarterly Bitcoin valuations during that time period that made Professor Liu rich, but Professor Liu does not remember what the valuations vector  $\vec{y}$  was during that time period. Knowing only  $\overrightarrow{\text{circcorr}}(\vec{y}, \vec{p}_A)$ , we will investigate a method to recover  $\vec{y}$ .

- (e) (4 points) Professor Liu recognizes that the matrix  $\mathbf{C}$  from part (d) is a **circulant matrix** and gives Sang Min a matrix called  $\mathbf{F}$  that diagonalizes  $\mathbf{C}$ , i.e.

$$\mathbf{C} = \mathbf{F}^T \mathbf{\Lambda} \mathbf{F},$$

where  $\mathbf{\Lambda}$  is a diagonal matrix.  $\mathbf{F}$  has a very special property: it is an orthogonal matrix that satisfies  $\mathbf{F}^{-1} = \mathbf{F}^T$ , i.e. the inverse of the matrix is equal to the transpose of the matrix. Sang Min does not remember how to invert  $\mathbf{C}$ , but he knows how to invert a diagonal matrix. **What is  $\vec{y}$  in terms of  $\overrightarrow{\text{circcorr}}(\vec{y}, \vec{p}_A)$ ,  $\mathbf{F}$ , and  $\mathbf{\Lambda}$ ?**

**Solution:** Using  $\mathbf{C} = \mathbf{F}^T \mathbf{\Lambda} \mathbf{F}$ , we have

$$\overrightarrow{\text{circcorr}}(\vec{y}, \vec{p}_A) = \mathbf{F}^T \mathbf{\Lambda} \mathbf{F} \vec{y}.$$

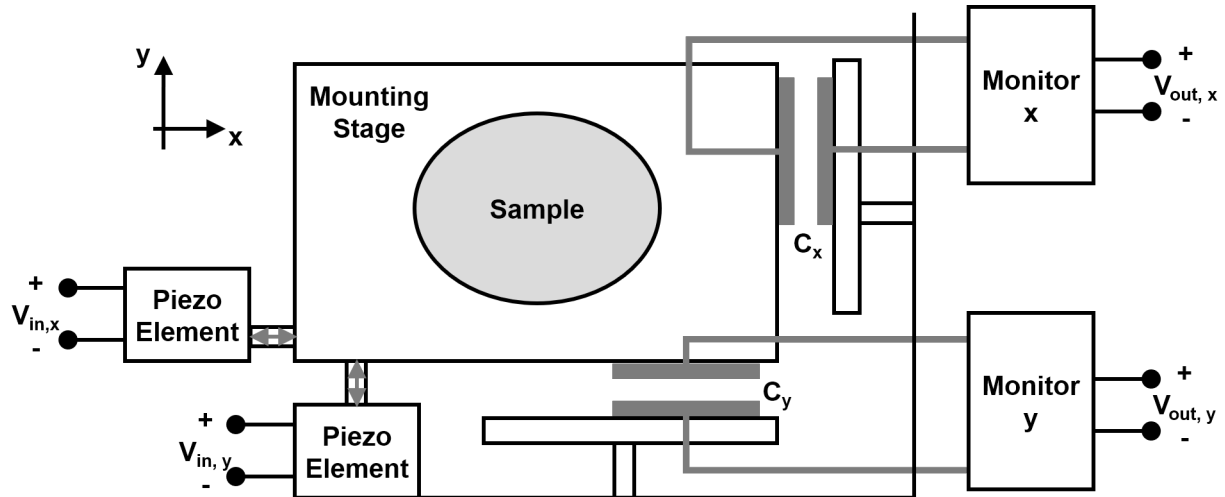
Using  $\mathbf{F}^{-1} = \mathbf{F}^T$  and assuming that  $\mathbf{C}$  is invertible here, we have

$$\begin{aligned} \mathbf{F} \overrightarrow{\text{circcorr}}(\vec{y}, \vec{p}_A) &= \mathbf{\Lambda} \mathbf{F} \vec{y} \\ \mathbf{\Lambda}^{-1} \mathbf{F} \overrightarrow{\text{circcorr}}(\vec{y}, \vec{p}_A) &= \mathbf{F} \vec{y} \\ \mathbf{F}^T \mathbf{\Lambda}^{-1} \mathbf{F} \overrightarrow{\text{circcorr}}(\vec{y}, \vec{p}_A) &= \vec{y}. \end{aligned}$$

PRINT your name and student ID: \_\_\_\_\_

### 7. 2D Piezo positioner (20 points)

We would like to modify the Piezo positioner, the device we covered in Midterm 2, for use in two dimensions. The overall structure of a 2D Piezo positioner is shown below.



The mounting stage moves within the  $x - y$  plane. There are two input voltages that control the location of the mounting stage along the  $x$ - and  $y$ - axes, respectively, as well as two capacitors. Each capacitance depends on the distance between the mounting stage and the measurement stage (in  $x$  and  $y$ , respectively).

Assume that there's a simple relationship between the input voltages and the outputs from the monitoring circuit, which is given as follows.

$$\begin{aligned} x_{\text{stage}} &= \alpha_x V_{\text{in},x}, & V_{\text{out},x} &= x_{\text{stage}} - x_{\text{desired}}, \\ y_{\text{stage}} &= \alpha_y V_{\text{in},y}, & V_{\text{out},y} &= y_{\text{stage}} - y_{\text{desired}}. \end{aligned}$$

We use the output voltages at time step  $i$  to update the input voltages at time step  $i + 1$  as shown by the equations below:

$$\begin{aligned} V_{\text{in},x}[i + 1] &= V_{\text{in},x}[i] - kV_{\text{out},x}[i], \\ V_{\text{in},y}[i + 1] &= V_{\text{in},y}[i] - kV_{\text{out},y}[i]. \end{aligned}$$

Let's say that we want to place the mounting stage at the origin ( $x_{\text{desired}} = y_{\text{desired}} = 0$ ), and define the following:

$$\begin{aligned} \vec{V}_{\text{in}}[i] &= \begin{bmatrix} V_{\text{in},x}[i] \\ V_{\text{in},y}[i] \end{bmatrix} \\ \vec{V}_{\text{out}}[i] &= \begin{bmatrix} V_{\text{out},x}[i] \\ V_{\text{out},y}[i] \end{bmatrix}. \end{aligned}$$

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- (a) (5 points) We would like to express the relationship between  $\vec{V}_{in}[i]$  and  $\vec{V}_{in}[i+1]$ . **Find A**, where

$$\vec{V}_{in}[i+1] = \mathbf{A}\vec{V}_{in}[i].$$

**Solution:**

$$\begin{aligned}\vec{V}_{out}[i] &= \begin{bmatrix} \alpha_x & 0 \\ 0 & \alpha_y \end{bmatrix} \vec{V}_{in}[i] - \begin{bmatrix} x_{desired} \\ y_{desired} \end{bmatrix} = \begin{bmatrix} \alpha_x & 0 \\ 0 & \alpha_y \end{bmatrix} \vec{V}_{in}[i] \\ \vec{V}_{in}[i+1] &= \vec{V}_{in}[i] - k\vec{V}_{out}[i] \\ &= \vec{V}_{in}[i] - k \begin{bmatrix} \alpha_x & 0 \\ 0 & \alpha_y \end{bmatrix} \vec{V}_{in}[i] \\ &= \left( I - k \begin{bmatrix} \alpha_x & 0 \\ 0 & \alpha_y \end{bmatrix} \right) \vec{V}_{in}[i] = \begin{bmatrix} 1 - k\alpha_x & 0 \\ 0 & 1 - k\alpha_y \end{bmatrix} \vec{V}_{in}[i]\end{aligned}$$

- (b) (2 points) The initial input voltages are given as a two-dimensional vector  $\vec{V}_{in}[0] = \begin{bmatrix} V_{in,x}[0] \\ V_{in,y}[0] \end{bmatrix}$ , whose entries are not zero. **Express  $\vec{V}_{in}[i]$  in terms of  $k$ ,  $\alpha_x$ ,  $\alpha_y$ ,  $i$  and  $\vec{V}_{in}[0]$ .**

**Solution:**

$$\vec{V}_{in}[i] = \begin{bmatrix} 1 - k\alpha_x & 0 \\ 0 & 1 - k\alpha_y \end{bmatrix} \vec{V}_{in}[i-1] = \dots = \begin{bmatrix} 1 - k\alpha_x & 0 \\ 0 & 1 - k\alpha_y \end{bmatrix}^i \vec{V}_{in}[0] = \begin{bmatrix} (1 - k\alpha_x)^i & 0 \\ 0 & (1 - k\alpha_y)^i \end{bmatrix} \vec{V}_{in}[0]$$

- (c) (4 points) Eventually, we want to ensure  $x_{stage} = x_{desired} = 0$  and  $y_{stage} = y_{desired} = 0$  as  $i \rightarrow \infty$ . **What conditions do  $\alpha_x$  and  $\alpha_y$  have to satisfy to guarantee this?** Leave your answers in terms of  $\alpha_x$ ,  $\alpha_y$ , and  $k$ .

**Solution:**

$$\begin{aligned}\lim_{i \rightarrow \infty} (1 - k\alpha_x)^i &= \lim_{i \rightarrow \infty} (1 - k\alpha_y)^i = 0 \\ -1 < 1 - k\alpha_x < 1, & \quad -1 < 1 - k\alpha_y < 1 \\ 0 < k\alpha_x < 2, & \quad 0 < k\alpha_y < 2\end{aligned}$$

- (d) (2 points) It turns out the previously known relationship between the input voltages and the outputs are incorrect, and the following is the updated relationship.

$$\begin{aligned}x_{stage} &= \alpha_{11}V_{in,x} + \alpha_{12}V_{in,y}, \quad V_{out,x} = x_{stage} - x_{desired}, \\ y_{stage} &= \alpha_{21}V_{in,x} + \alpha_{22}V_{in,y}, \quad V_{out,y} = y_{stage} - y_{desired}.\end{aligned}$$

**Find the matrix B such that  $\vec{V}_{in}[i+1] = \mathbf{B}\vec{V}_{in}[i]$ .**

**Solution:**

$$\begin{aligned}\vec{V}_{out}[i] &= \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \vec{V}_{in}[i] \\ \vec{V}_{in}[i+1] &= \vec{V}_{in}[i] - k\vec{V}_{out}[i] \\ &= \left( I - k \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \right) \vec{V}_{in}[i] = \begin{bmatrix} 1 - k\alpha_{11} & -k\alpha_{12} \\ -k\alpha_{21} & 1 - k\alpha_{22} \end{bmatrix} \vec{V}_{in}[i]\end{aligned}$$

- (e) (7 points) Suppose that  $k = 0.1$ ,  $\alpha_{11} = 4$ ,  $\alpha_{12} = 1$ ,  $\alpha_{21} = -2$ , and  $\alpha_{22} = 7$ . **Does this system converge to  $x_{\text{stage}} = y_{\text{stage}} = x_{\text{desired}} = y_{\text{desired}} = 0$ ?** (*Hint: use diagonalization.*)

**Solution:**

$$\vec{V}_{\text{in}}[i+1] = \begin{bmatrix} 1 - k\alpha_{11} & -k\alpha_{12} \\ -k\alpha_{21} & 1 - k\alpha_{22} \end{bmatrix} \vec{V}_{\text{in}}[i] = \begin{bmatrix} 1 - 0.4 & -0.1 \\ 0.2 & 1 - 0.7 \end{bmatrix} \vec{V}_{\text{in}}[i] = \begin{bmatrix} 0.6 & -0.1 \\ 0.2 & 0.3 \end{bmatrix} \vec{V}_{\text{in}}[i]$$

Through diagonalization,

$$\begin{bmatrix} 0.6 & -0.1 \\ 0.2 & 0.3 \end{bmatrix} = P\Lambda P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

As a result,

$$\begin{aligned} \vec{V}_{\text{in}}[i] &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \vec{V}_{\text{in}}[i-1] \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.5^2 & 0 \\ 0 & 0.4^2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \vec{V}_{\text{in}}[i-2] \\ &= \dots = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.5^i & 0 \\ 0 & 0.4^i \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \vec{V}_{\text{in}}[0] \end{aligned}$$

Since  $\lim_{i \rightarrow \infty} 0.5^i = \lim_{i \rightarrow \infty} 0.4^i = 0$ , input voltages are going to converge to zero, and from the relationship between the stage position and input voltages, we can conclude that the system will converge to the origin.



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### 8. Word Embeddings (12 points)

In Natural Language Processing, we often represent words by vectors that correspond to their meanings. These vectors are called word embeddings, and they have some nice properties. One property is that if we have the vector  $\vec{x}_{\text{dog}}$  representing the word “dog” and the vector  $\vec{x}_{\text{house}}$  representing the word “house,” then the vector for “doghouse” is some linear combination

$$\vec{x}_{\text{doghouse}} = \alpha\vec{x}_{\text{dog}} + \beta\vec{x}_{\text{house}}.$$

We will call words like “dog” and “house” **simple words**, while words like “doghouse” are **compound words**. The embedding for a compound word can always be expressed as the linear combination of two simple word embeddings.

(a) (4 points) My friend has a matrix  $\mathbf{A}$  where each column  $\vec{a}_i$  is a simple word embedding and nonzero:

$$\mathbf{A} = \begin{bmatrix} | & | & \dots & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & \dots & | \end{bmatrix}.$$

Suppose  $\vec{x} = \alpha\vec{a}_u + \beta\vec{a}_v$  is a compound word embedding composed of two words from  $\mathbf{A}$ , where  $\alpha, \beta \neq 0$ . Also, suppose that the columns of  $\mathbf{A}$  are orthogonal, so for  $i \neq j$ ,  $\vec{a}_i^T \vec{a}_j = 0$ .

**Then, for which indices  $i$  will the inner product  $\vec{x}^T \vec{a}_i$  be non-zero? Justify your answer.**

**Solution:** The inner product is nonzero for  $i = u$  or  $v$ . We can see this by expanding the inner product linearly:

$$\vec{x}^T \vec{a}_i = \alpha\vec{a}_u^T \vec{a}_i + \beta\vec{a}_v^T \vec{a}_i.$$

Then, if  $i \neq u$  and  $i \neq v$ , then both terms are zero. If  $i = u$ , we get  $\alpha\|\vec{a}_u\|$ , and if  $i = v$ , we get  $\beta\|\vec{a}_v\|$ .

(b) (4 points) Let

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & -1 & 3 \\ -2 & 2 & 3 \\ 6 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

be a matrix whose columns are simple word embeddings. Note that the columns are orthogonal. **Which two columns make up the following compound word embedding?**

$$\begin{bmatrix} 5 \\ 1 \\ 5 \\ -1 \\ 12 \\ -1 \end{bmatrix},$$

**Justify your answer.**

**Solution:** It is made up of the first and third columns. We can see this by taking the inner product with the second column, which gives us zero.

(c) (4 points) Suppose  $\mathbf{C}$  is an  $n \times n$  matrix with columns  $\vec{c}_1, \dots, \vec{c}_n$ , where we have

$$\vec{c}_i^T \vec{c}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

**Prove that the inverse of  $\mathbf{C}$  is  $\mathbf{C}^T$ .**

**Solution:** It suffices to show that  $\mathbf{C}^T \mathbf{C} = \mathbf{I}$ . The  $i, j$ -th entry of  $\mathbf{C}^T \mathbf{C}$  is the inner product  $\vec{c}_i^T \vec{c}_j$ , which is one for  $i = j$  and zero otherwise. Therefore,  $\mathbf{C}^T \mathbf{C}$  has ones on the diagonal and zeros otherwise.

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### 9. Pineapple Pioneers: Part 1 (20 points)

Your TAs Nick and Linda are big fans of pineapple on their pizza. Unfortunately, they don't have enough money to buy a nice pineapple pizza, so they decide to make one instead. They decide to grow a pineapple plant in the dirt outside of Cory Hall, but the harsh environment makes it hard for the plant to grow! Bugs keep trying to eat their plant, so they decide to make an alarm system that will sound a horn and scare them away, but they need your help.

Note:  $m$  (milli) is  $10^{-3}$ ,  $\mu$  (micro) is  $10^{-6}$  and  $n$  (nano) is  $10^{-9}$ .

- (a) (6 points) You tell them a capacitive sensor might work. With your immense 16A knowledge, you construct a special capacitor  $C_{\text{special}}$  that will change capacitance values when a bug touches it, and place them all around the growing plant. The behavior of  $C_{\text{special}}$  is such that:

$$C_{\text{special}} = \begin{cases} C_1 & \text{Bug is present} \\ C_2 & \text{No bug is present} \end{cases}$$

You decide to drive this capacitor with a current source of value  $I_S = 1\mu\text{A}$  as shown below.

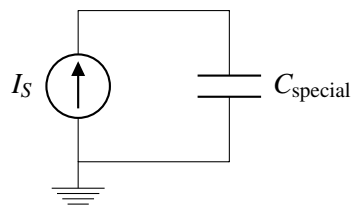


Figure 9.1: Capacitor charging circuit

You charge the capacitor for 1 second and record the voltage across it. You then discharge the capacitor and place a bug on it. You again allow it to charge for 1 second. The following behavior is observed:

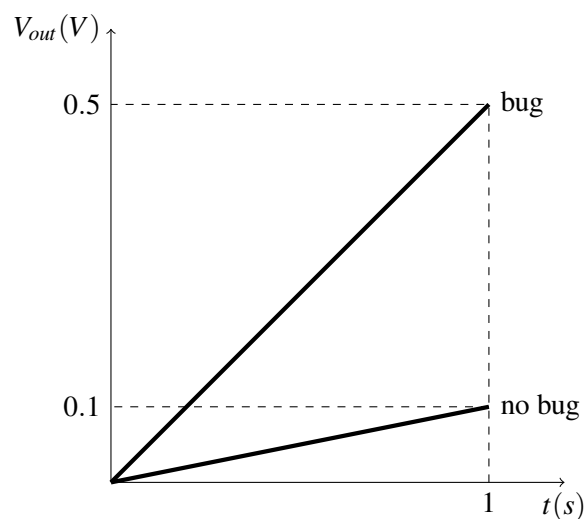


Figure 9.2: Capacitor behavior

**Assume the capacitor is initially uncharged, and is fully discharged in between the two tests. Find values for  $C_1$  and  $C_2$  that give the behavior in Figure 9.2.**

**Solution:** The voltage across a capacitor driven by a constant current is

$$V(t) = \frac{I_s}{C_{\text{special}}}t + V(0)$$

since the capacitor is initially uncharged, this reduces to

$$V(t) = \frac{I_s}{C_{\text{special}}}t$$

When a bug is present, after 1 second we have

$$V(1) = 0.5V = \frac{1\mu A}{C_1} \times 1$$

which gives  $C_1 = 2\mu F$

Without the bug present we have

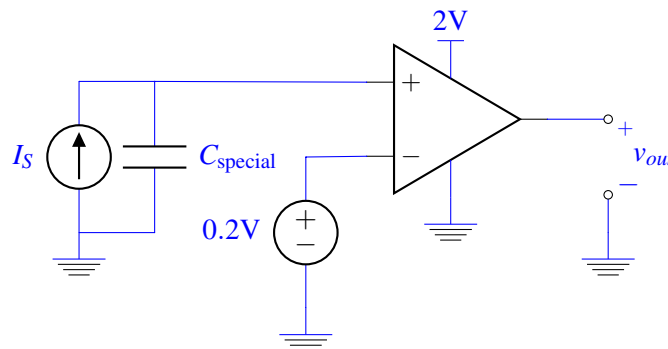
$$0.1 = \frac{1\mu A}{C_2} \times 1$$

So  $C_2 = 10\mu F$

- (b) (8 points) Regardless of your answer to part (a), assume  $C_1 = 5\text{nF}$  and  $C_2 = 12\text{nF}$ . Nick and Linda want to only check the voltage after some time, and output either 2V or 0V depending on if a bug is present or not to make things easier. You find that you have one op-amp, one voltage source of value 0.2V, and one voltage source of 2V.

**Using the circuit from part (a) and the provided components only, design a circuit that after some time  $t$ , outputs 2V if a bug is present, and 0V if not. Also, determine the value of  $t$  after which the output is valid. Assume the capacitor is initially uncharged and that we are only checking the circuit once (i.e., don't worry about resetting the capacitor's charge). You must label the supplies on the op amp.**

**Solution:** We want either 2V or 0V, which implies we should use a comparator.



To output 2V, we need  $u^+$  to be greater than 0.2V. If no bug is present, the capacitor voltage will increase linearly from 0V (initially uncharged) indefinitely. We need to find the time where the capacitor voltage surpasses 0.2V. Note that no current enters the op amp so, the all of it goes through the

capacitor.

$$\begin{aligned}
 V(t) &= \frac{I_s}{C}t + V(0) \\
 0.2 &= \frac{1\mu A}{C_1}t + 0 \\
 \frac{0.2 * C_1}{10^{-6}} &= t \\
 \frac{2 * 5 * 10^{-1} * 10^{-9}}{10^{-6}} &= t \\
 t &= 1ms
 \end{aligned}$$

(c) (5 points) While testing the circuit, your current source breaks! You manage to find another one, but it varies with time. The new circuit and behavior are shown below.

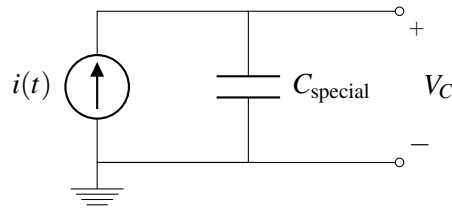
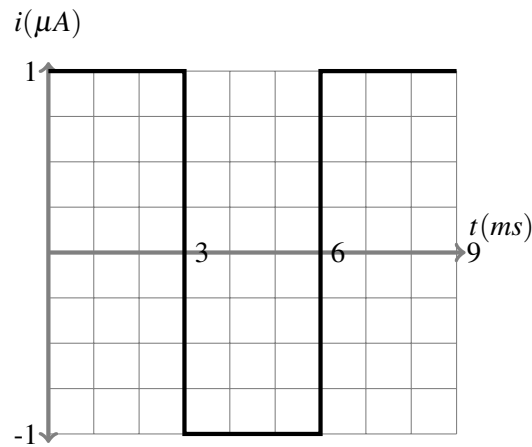
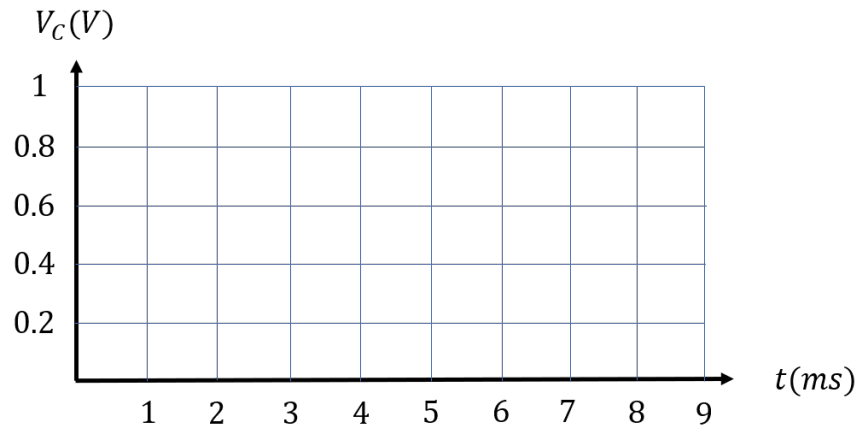


Figure 9.3: New current source



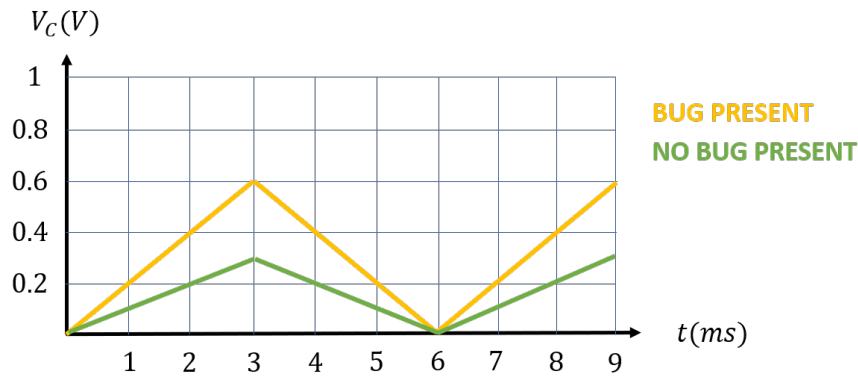
**On the graph below, plot the voltage across the capacitor ( $V_C$ ) as a function of time for the cases where a bug is present and where no bug is present. Clearly label which is which. Assume  $C_1 = 5nF$  and  $C_2 = 12nF$ . Assume the capacitor is initially uncharged.**



**Solution:** Begin by plugging into the same equation as before.

$$V(t) = \frac{I_s}{C}t + V(0)$$

At a time of 3ms, this results in 0.6V with  $C_1$  and 0.25V with  $C_2$ . We know the increase is linear so we get the following plot:



(d) (1 point) Does pineapple belong on pizza? *You can put anything you want or nothing at all and you will get the points for this part.*

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### 10. Pineapple Pioneers: Part 2 (30 points)

Earlier in the exam, Nick and Linda tried to grow a pineapple plant outside Cory Hall, but found the environment to be too difficult to grow anything in. To increase their yield, they decide to ask Ryan for suggestions. He suggests that they move their operation to South America where the environment is better. Since it's so far away, Nick and Linda decide to monitor the plant wirelessly by sending data from various sensors with an antenna.

Back in Berkeley, with the help of Ryan, they come up with the following setup to retrieve the information and process it. The first device is an antenna, which you don't need to understand for this problem.

Note:  $m$  (milli) is  $10^{-3}$  and  $n$  (nano) is  $10^{-9}$ .

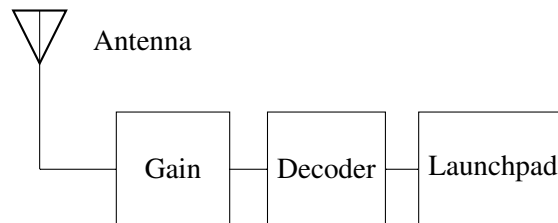


Figure 10.1: Receiver block diagram

Unfortunately, the best the trio can come up with is a block diagram, and they need your help implementing these blocks.

- (a) (4 points) Let's first start by looking at the antenna. The Thevenin equivalent of the antenna is shown in Figure 10.2.

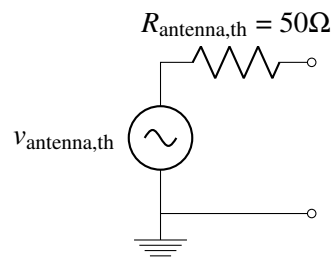


Figure 10.2: Antenna Thevenin equivalent

The resistance of the load attached to the antenna is very important for many reasons. One reason is to maximize power transfer. **If we were to attach a load resistor  $R_L$  to the open port on this circuit, what should the resistance be to have maximum power dissipated in  $R_L$ ? Justify your answer.**

**Solution:** We've seen this on homework before. To find the maximum, we will want to take the

derivative of the power expression with respect to  $R_L$  and set it to 0.

$$P = \frac{V^2}{R_L}, V = v_{\text{antenna,th}} \frac{R_L}{R_{th} + R_L}$$

$$P = \frac{R_L}{(R_{th} + R_L)^2} V_{\text{antenna,th}}^2$$

$$\frac{dP}{dR_L} = v_{\text{antenna,th}}^2 \frac{(R_L + R_{th})^2 - 2R_L(R_L + R_{th})}{(R_L + R_{th})^4} = 0$$

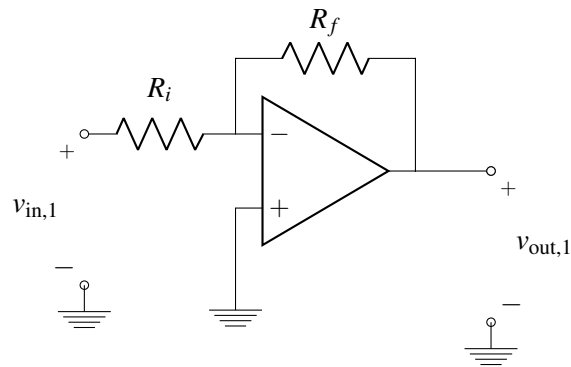
$$(R_L + R_{th})^2 - 2R_L(R_L + R_{th}) = 0$$

$$R_L^2 + 2R_LR_{th} + R_{th}^2 - 2R_L^2 - 2R_LR_{th} = 0$$

$$R_L^2 = R_{th}^2$$

$$R_L = R_{th}$$

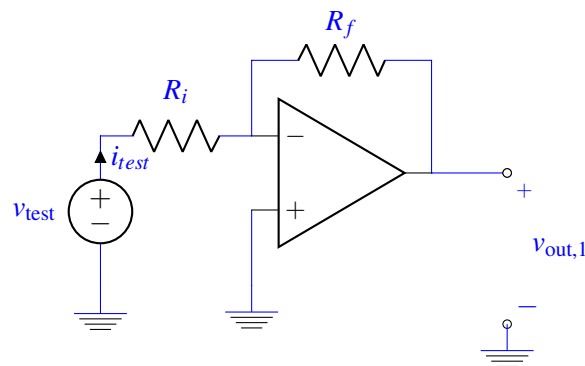
- (b) (4 points) Let's now look at the "Gain" block. The signal will be significantly weakened when traveling through the air, so we need to apply gain to what comes out of the antenna or our Launchpad won't be able to tell what any of the data means. Ryan, Nick, and Linda ask their local wireless experts for a suggestion, and they suggest the following circuit:



They claim that you can make this circuit "look like" whatever resistance you want. To test their theory, **find the Thevenin resistance with respect to the port  $v_{in,1}$ .**

*Hint: use the  $V_{test}/I_{test}$  method.*

**Solution:** Follow the hint and find the Thevenin resistance with  $V_{test}/I_{test}$ .





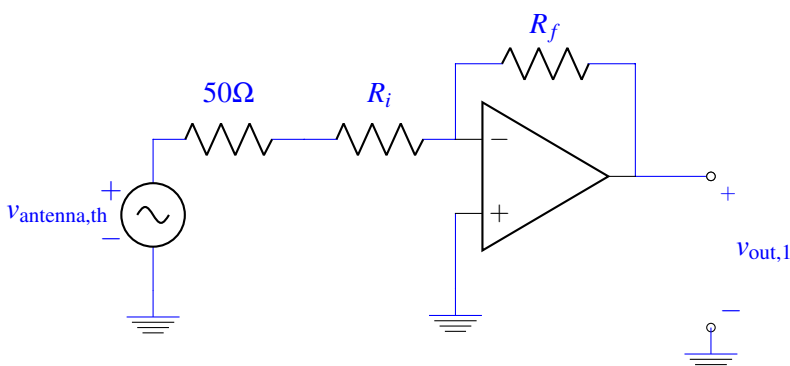
We know that  $V_{R_i} = i_{test}R_i$ , and we also know the op amp is in negative feedback. Thus,  $u^- = 0$ . So

$$\begin{aligned} v_{test} - u^- &= i_{test}R_i \\ v_{test} &= i_{test}R_i \\ \frac{v_{test}}{i_{test}} &= R_i \end{aligned}$$

So the Thevenin resistance is just  $R_i$ .

- (c) (4 points) We now want to finish this circuit. Suppose that we need an overall gain with a magnitude of 1000 (i.e.  $|\frac{v_{out,1}}{v_{antenna,th}}| = 1000$ ). **Choose values of  $R_f$  and  $R_i$  for the circuit in (b) to achieve the required gain and to have a Thevenin resistance at  $v_{in,1}$  for maximum power.**

**Solution:** From part (a), we know that  $R_{th} = R_L = 50\Omega$ . From part (b) we know  $R_L = R_i$ . Thus we should have  $R_i = 50\Omega$ . When we attach the antenna Thevenin circuit, we will get  $R_{th}$  in series with  $R_i$  like below.

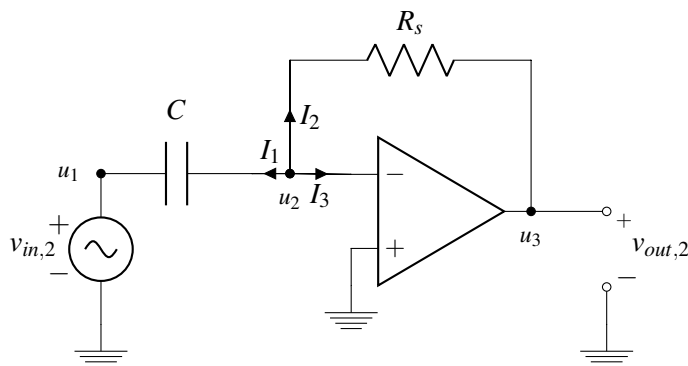


So we have from the inverting amplifier equation and knowing  $R_i = 50\Omega$ ,

$$\begin{aligned} v_{out,1} &= -\frac{R_f}{R_i + 50} v_{antenna,th} \\ v_{out,1} &= -\frac{R_f}{50 + 50} v_{antenna,th} \\ \frac{v_{out,1}}{v_{antenna,th}} &= -\frac{R_f}{100} \end{aligned}$$

Thus for a gain of 1000 in magnitude,  $R_f = 100k\Omega$ .

- (d) (8 points) Now that we have a signal with a large amplitude, we need to decode it. The trio decides to use FM (like the radios in cars) to send the data, and now they have to decode it. FM is quite complicated, so they enlist EE123 students for help. The EE123 students will do most of the work if Nick, Ryan, and Linda can do one thing first. FM is encoded in such a way that you need to take the derivative of the signal to extract the information. The trio asks Professor Stojanovic for help and he suggests the following circuit:



Let's analyze it in steps.

- i. (2 points) **Write the KCL expression at node  $u_2$  in terms of  $I_1, I_2$  and  $I_3$  only.**
- ii. (6 points) **Substitute IV relationships for the components and apply the Golden Rules. Solve for  $v_{out,2}$  as a function of  $R, C$  and  $v_{in,2}$ . You may assume negative feedback.**

**Solution:**

- i. KCL:  $I_1 + I_2 + I_3 = 0$  since all currents are leaving, they all have the same sign.
- ii. Plugging in the Current-Voltage relationships we get

$$I_1 + I_2 + I_3 = 0$$

$$C \frac{d(u_2 - v_{in,2})}{dt} + \frac{u_2 - v_{out,2}}{R_s} + I_3 = 0$$

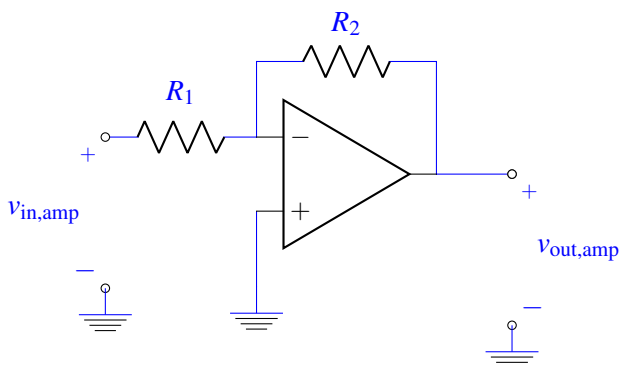
from the golden rules, we know that  $u_2 = 0$  and  $I_3 = 0$  so

$$-C \frac{d(v_{in,2})}{dt} - \frac{v_{out,2}}{R_s} = 0$$

$$v_{out,2} = -R_s C \frac{dv_{in,2}}{dt}$$

- (e) (6 points) Unfortunately, the above circuit also scales our output by some factor. We **want** an output that is exactly  $v_{out} = 1 \text{ s} \times \frac{dv_{in,2}}{dt}$ . Note that the scaling factor must have units of time for the expression to work out. **Design a circuit whose input is  $v_{out,2}$  from (d) and produces an output  $v_{out} = 1 \text{ s} \times \frac{dv_{in,2}}{dt}$ . You may use one op amp and two resistors. The resistors must be given values in terms of  $C$  and  $R_s$  from part (d).**

**Solution:** From the previous part, we know the output is almost correct, but is scaled by  $-R_s C$ . We can use an inverting amplifier to undo this.



$$v_{out,amp} = -\frac{R_2}{R_1} v_{in,amp}$$

By putting the output of the differentiator into the input of the amplifier, we get this expression.

$$v_{out,amp} = \frac{R_2}{R_1} R_s C \frac{dv_{in,2}}{dt}$$

Since we want the output the only the differentiation, the amplification should cancel the  $R_s C$ :

$$\frac{R_2}{R_1} R_s C = 1 \text{ sec}$$

Any choice of  $R_1$  and  $R_2$  that satisfy this works. One example is  $R_2 = 1\text{k}\Omega$  and  $R_1 = 1000R_s C \frac{\Omega}{\text{sec}}$ .

- (f) (4 points) Now assume that every block in the diagram (repeated below) has been perfectly implemented.

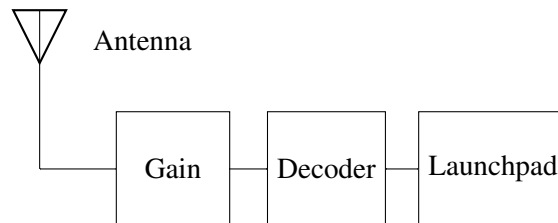
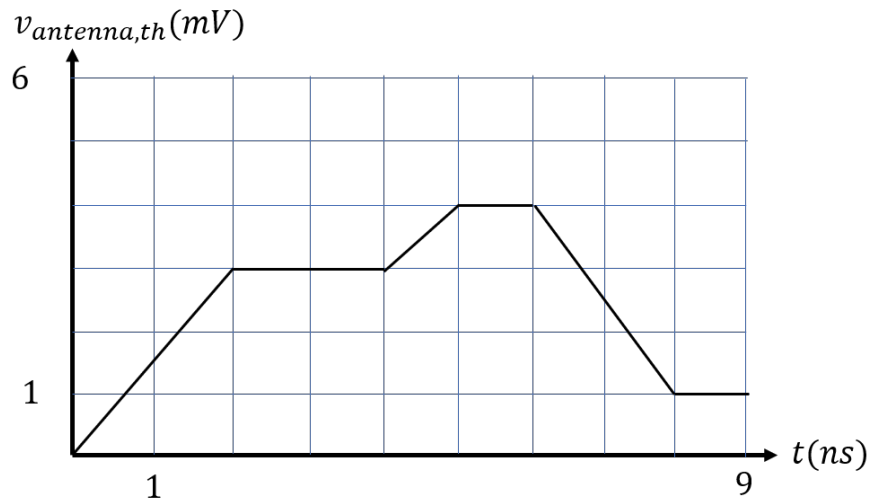
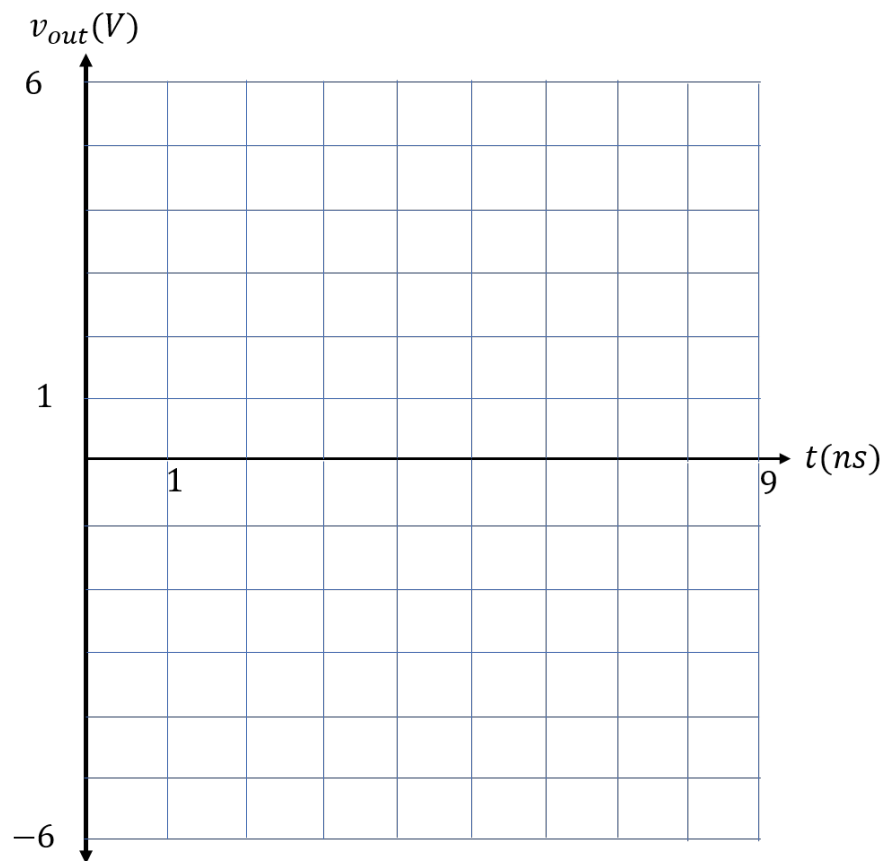


Figure 10.3: Receiver block diagram, repeated

Regardless of your previous answers, assume that Gain = -1000 and the decoder output is  $v_{out} = 1 \text{ ns} \times \frac{dv_{in}}{dt}$  (nanoseconds because of the time scale of the graph). You are given the input from the antenna as a function of time below. **On the empty graph, plot the output of the entire receiver (the output of part (e), the decoder).**

*Note: pay attention to the scales on the graphs.*





**Solution:** Overall through the system we have a gain of -1000 and a derivative. Thus we should take the slope of each part and scale it by -1000.

