EECS 16A Designing Information Devices and Systems I
Summer 2017 D. Aranki, F. Maksimovic, V. Swamy Midterm 1

Exam Location: 2050 VLSB

PRINT your student ID: ____

PRINT AND SIGN your name: ____________________, ____________________ ____________________
(last name) (first name) (signature)

PRINT your Unix account login: ee16a-_____________

PRINT your discussion section and GSI(s) (the one you attend): ____

Name and SID of the person to your left: ____

Name and SID of the person to your right: ____

Name and SID of the person in front of you: ____

Name and SID of the person behind you: ____

1. What did you do on the Fourth of July? (1 point)

2. What activity do you really enjoy? (1 point)

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

3. Mechanical Matrix (12 points)

Consider the matrix **A** below:

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) (4 points) **Calculate $\mathbf{A}^2$.**

Solution:

$$\mathbf{A}^2 = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -3 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) (2 points) **What is $\det(\mathbf{A})$?**

Solution:

Since **A** is upper triangular, the determinant of **A** is equal to the product of its diagonal entries.

$$\det(\mathbf{A}) = 2 \cdot (-2) \cdot 1 = -4$$

(c) (6 points) **Calculate $\mathbf{A}^{-1}$, the inverse of A.**

Solution:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \leftarrow R_1 + R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 1 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_1 \leftarrow \frac{1}{2}R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_2 \leftarrow -\frac{1}{2}R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

PRINT your name and student ID: ____

4. You Have Null Idea (12 points)

Consider the matrix $\mathbf{A}$ below:

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 5 & 1 \\ 6 & 3 & 3 \end{bmatrix}$$

(a) (2 points) **Is A invertible? Justify your answer.**

Solution:

You can visually inspect that column 1 of $\mathbf{A}$ is 2-times column 3 of $\mathbf{A}$. Therefore, the two columns are linearly dependent, and $\mathbf{A}$ is not invertible by the Invertible Matrix Theorem.

If you don't observe the dependence in the columns immediately, you can alternatively solve this part using Gaussian elimination.

$$\begin{bmatrix} 4 & 1 & 2 \\ 2 & 5 & 1 \\ 6 & 3 & 3 \end{bmatrix}$$

Switching row 1 and row 2 gives

$$\begin{bmatrix} 2 & 5 & 1 \\ 4 & 1 & 2 \\ 6 & 3 & 3 \end{bmatrix}$$

Dividing row 3 by 3 gives

$$\begin{bmatrix} 2 & 5 & 1 \\ 4 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

Subtracting row 1 from row 3 gives

$$\begin{bmatrix} 2 & 5 & 1 \\ 4 & 1 & 2 \\ 0 & -4 & 0 \end{bmatrix}$$

Dividing row 3 by -4 gives

$$\begin{bmatrix} 2 & 5 & 1 \\ 4 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

Subtracting row 3 from row 2 gives

$$\begin{bmatrix} 2 & 5 & 1 \\ 4 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

Subtracting 5 times row 3 from row 1 gives

$$\begin{bmatrix} 2 & 0 & 1 \\ 4 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

Subtracting 2 times row 1 form row 2 gives

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Since there is a row of all zeros, the matrix is not invertible.

(b) (4 points) **Find a basis for the column space of A.**

Solution:

The first two columns of the reduced row echelon form of $\mathbf{A}$ have pivots. Therefore, the first two columns of $\mathbf{A}$ constitute a basis for the column space of $\mathbf{A}$. That is, $\left\{ \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}$ is a basis for $\text{Col}(\mathbf{A})$.

A copy of the matrix for reference: $\mathbf{A} = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 5 & 1 \\ 6 & 3 & 3 \end{bmatrix}$

(c) (6 points) **Find a basis for the null space of A.**

Solution:

From part (a), we know the row echelon form of $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 5 & 1 \\ 6 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The third column has no pivot, so x_3 is a free variable. From the second equation, we know that $x_2 = 0$. From the first equation, we know that $x_1 = -\frac{1}{2}x_3$.

Therefore, the null space is given by

$$\vec{x} = x_3 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}, x_3 \in \mathbb{R}.$$

A basis for the null space of $\mathbf{A}$ would be $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$.

5. Diagonalization (8 points)

Consider the matrix **A** below:

$$\mathbf{A} = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$$

If the matrix A is diagonalizable as $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$, write out the matrices V, Λ, and $\mathbf{V}^{-1}$ explicitly. If the matrix A is not diagonalizable, explain why.

Solution:

Start by calculating the eigenvalues:

$$\begin{aligned} \left| \begin{bmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{bmatrix} \right| &= 0 \\ (6-\lambda)(3-\lambda)+2 &= 0 \\ \lambda^2 - 9\lambda + 20 &= 0 \\ (\lambda-5)(\lambda-4) &= 0 \\ \lambda &= 4,5 \end{aligned}$$

Since the eigenvalues are distinct, we will have 2 linearly independent eigenvectors, which means that **A** is diagonalizable. Therefore, we can write $\mathbf{\Lambda} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$.

We now calculate the corresponding the eigenvectors (matrices **V** and $\mathbf{V}^{-1}$):

For $\lambda = 4$:

$$\begin{aligned} \begin{bmatrix} 6-4 & -1 \\ 2 & 3-4 \end{bmatrix} \vec{x} &= \vec{0} \\ \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \vec{x} &= \vec{0} \\ \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \vec{x} &= \vec{0} \\ \vec{x} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

For $\lambda = 5$:

$$\begin{aligned} \begin{bmatrix} 6-5 & -1 \\ 2 & 3-5 \end{bmatrix} \vec{x} &= \vec{0} \\ \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \vec{x} &= \vec{0} \\ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \vec{x} &= \vec{0} \\ \vec{x} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore, $\mathbf{V} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, and using the 2×2 inverse formula from lecture, we know that $\mathbf{V}^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$.

6. Vector Space (16 points)

Consider $\mathbb{R}^3$, a vector space with the usual operations of vector addition and scalar vector multiplication. Let S be the following set of vectors from $\mathbb{R}^3$:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

(a) (2 points) **Is the set S linearly independent or is it linearly dependent? Justify your answer.**

Solution:

S is linearly independent because the vectors aren't a scaled version of one another.

(b) (6 points) Let $\mathbb{W} = \text{span}(S)$. **Show that $\mathbb{W}$ forms a subspace of $\mathbb{R}^3$.**

Solution:

Let's give names to the vectors in S: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Then

$$\mathbb{W} = \{\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \subset \mathbb{R}^3.$$

Closure:

i. Scaling: If $\vec{w} \in \mathbb{W}$ and $\beta \in \mathbb{R}$, then $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2$ and therefore $\beta\vec{w} = \beta\alpha_1\vec{v}_1 + \beta\alpha_2\vec{v}_2$, which means $\beta\vec{w} \in \mathbb{W}$.

ii. Additivity: If $\vec{w}, \vec{y} \in \mathbb{W}$, then $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2$ and $\vec{y} = \beta_1\vec{v}_1 + \beta_2\vec{v}_2$. Since $\vec{w} + \vec{y} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \beta_1\vec{v}_1 + \beta_2\vec{v}_2 = (\alpha_1 + \beta_1)\vec{v}_1 + (\alpha_2 + \beta_2)\vec{v}_2$, which means $\vec{w} + \vec{y} \in \mathbb{W}$.

Zero: $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 \in \mathbb{W}$.

We showed the three properties that prove that $\mathbb{W}$ is a subspace of $\mathbb{R}^3$.

(c) (6 points) Let $\vec{v} = \begin{bmatrix} 2m \\ m+3 \\ 0 \end{bmatrix}, m \in \mathbb{R}$.

For what value(s) of m is $\vec{v} \in \text{span}(S)$?

Solution:

$\vec{v} = \begin{bmatrix} 2m \\ m+3 \\ 0 \end{bmatrix}$ is in $\text{span}(S) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ if we can write, for some α_1, α_2 and m:

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2m \\ m+3 \\ 0 \end{bmatrix} \tag{1}$$

We solve for α_1, α_2 and m by rewriting the system of equations 1 into

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - m \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}.$$

By writing this system in matrix form, we get:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 3 \\ 1 & 2 & 0 & 0 \end{array}\right] \overset{R_3 \leftarrow R_3 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & 2 & 0 \end{array}\right] \overset{R_3 \leftarrow R_3 - R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 3 & -3 \end{array}\right]$$

At this point we know that $m = -1$ and we can stop. However, if we want to solve for α_1 and α_2 to verify our solution, we continue:

$$\overset{R_3 \leftarrow \frac{1}{3} R_3}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \overset{R_1 \leftarrow R_1 + 2R_3}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\overset{R_2 \leftarrow R_2 + R_3}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \overset{R_1 \leftarrow R_1 - R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Therefore, $\alpha_1 = -4, \alpha_2 = 2$, and $m = -1$.

If we verify: $-4 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2m \\ m+3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$, so our solution is correct.

(d) (2 points) We define a new set

$$S_{\text{new}} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2m \\ m+3 \\ 0 \end{bmatrix} \right\}$$

For what value(s) of m will S_{new} be a valid basis for $\mathbb{R}^3$?

Solution:

Because S is linearly independent, the augmented set S_{new} will be a basis as long as $\vec{v}$ is not in the span of the original S (that is, as long as $\begin{bmatrix} 2m \\ m+3 \\ 0 \end{bmatrix}$ is linearly independent of the elements of S). This is to say that as long as $m \neq -1$, $\vec{v}$ will complete S_{new} to become a valid basis for $\mathbb{R}^3$.

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7. Matrix Trix (10 points)

Let $n \in \{1,2,\dots\}$ be a positive integer. We define the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

Given a vector $\vec{x} \in \mathbb{R}^n$, $T(\vec{x})$ reverses the order of the n components in the input vector.

Explicitly, $T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) \triangleq \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{bmatrix}$.

(a) (2 points) Let's start with an example where $n = 4$. Suppose we have a vector $\vec{a} \in \mathbb{R}^4$, where $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$.

Then, if we apply the transformation T, we reverse the order of the entries, so $T(\vec{a}) = \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \end{bmatrix}$. **Find the matrix representation $\mathbf{A}$ of the transformation T, such that $T(\vec{a}) = \mathbf{A}\vec{a}$.**

Solution:

$$\mathbf{A} = \begin{bmatrix} | & | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) & T(\vec{e}_4) \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(b) (1 point) We now go back to the general case. Let $n \in \{1,2,\dots\}$ be a positive integer. Again, $T : \mathbb{R}^n \to \mathbb{R}^n$ is defined, as described above, as $T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{bmatrix}$. Let $\mathbf{B}$ be the matrix representation of the transformation T. **For any $\vec{x} \in \mathbb{R}^n$, write $\mathbf{B}^2\vec{x}$ in terms of $\vec{x}$.**

Solution:

$$\mathbf{B}^2\vec{x} = T(T(\vec{x})) = \vec{x}$$

(c) (7 points) **Show that if λ is an eigenvalue of the matrix B, then either $\lambda = 1$ or $\lambda = -1$ (that is, λ can't have any value other than 1 or -1).**

Hint: You don't need to write out the matrix **B** explicitly.

Solution:

Suppose that λ is an eigenvalue of **B** with the corresponding eigenvector $\vec{x}$.

$$\mathbf{B}\vec{x} = \lambda\vec{x}$$

Left-multiplying both sides by **B** gives

$$\mathbf{B}^2\vec{x} = \lambda\mathbf{B}\vec{x} = \lambda^2\vec{x}.$$

We know from part (b) that $\mathbf{B}^2\vec{x} = \vec{x}$, so we equate the RHS of these equations.

$$\lambda^2\vec{x} = \vec{x}$$

$$(\lambda^2 - 1)\vec{x} = \vec{0}$$

Since the eigenvector $\vec{x}$ is a non-zero vector, we get that $\lambda^2 - 1 = 0$, which implies $\lambda^2 = 1 \implies \lambda = \pm 1$.

8. Aw Snap, I'm All Ears (15 points)

One day while Daniel, Vasuki, and Fil are playing with the filters on Snapchat, they realize that what they're teaching in EE16A is sufficient to understand how the animated filters can be adjusted to match different facial structures.

Each animated face filter needs three anchor points to place the filter. The default anchor points are $\vec{p}_1 = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$, $\vec{p}_2 = \begin{bmatrix} 15 \\ 15 \end{bmatrix}$, and $\vec{p}_3 = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$. See Figure 8.1 for an example placement.

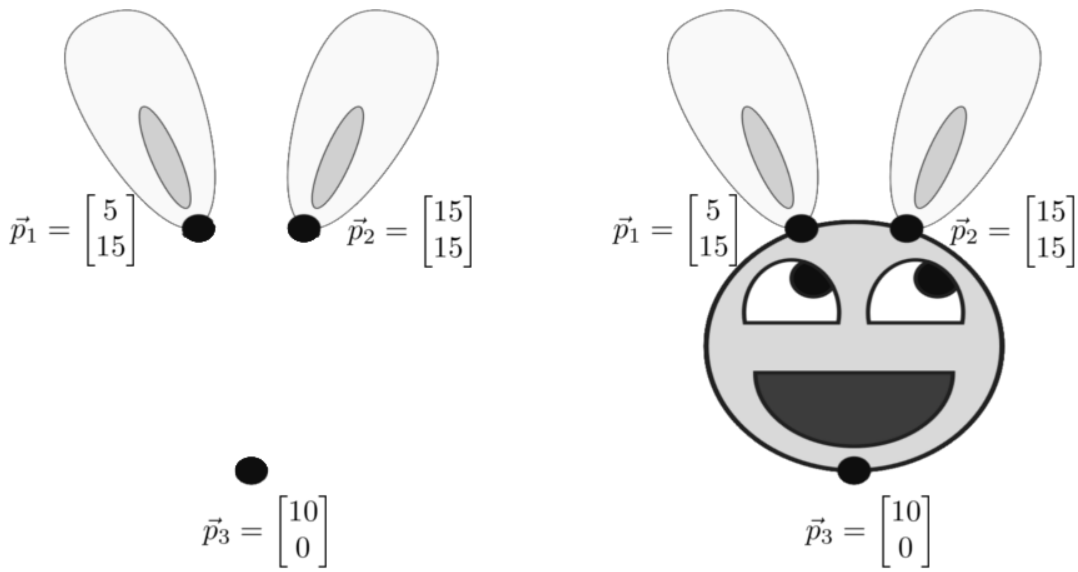

Figure 8.1: Left: bunny ears filter with default anchor points. Right: bunny ears filter placed onto a face using the default anchor points.

However, these default anchor points don't always work for all users. Now, it's up to you to line up each filter's anchor points to the user's face's matching anchor points.

The parts for this problem start on the next page.

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(a) (3 points) Daniel's anchor points are $\vec{d}_1 = \begin{bmatrix} 12.5 \\ 45 \end{bmatrix}, \vec{d}_2 = \begin{bmatrix} 37.5 \\ 45 \end{bmatrix}$, and $\vec{d}_3 = \begin{bmatrix} 25 \\ 0 \end{bmatrix}$ (see Figure 8.2). **Derive a linear transformation matrix A that appropriately scales the default anchor points to fit Daniel's face. Precisely, the transformation should achieve $\mathbf{A}\vec{p}_i = \vec{d}_i$ for all $i \in \{1,2,3\}$. It should take the form of $\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, a, b \in \mathbb{R}$. If this is not possible, prove that it is not possible.**

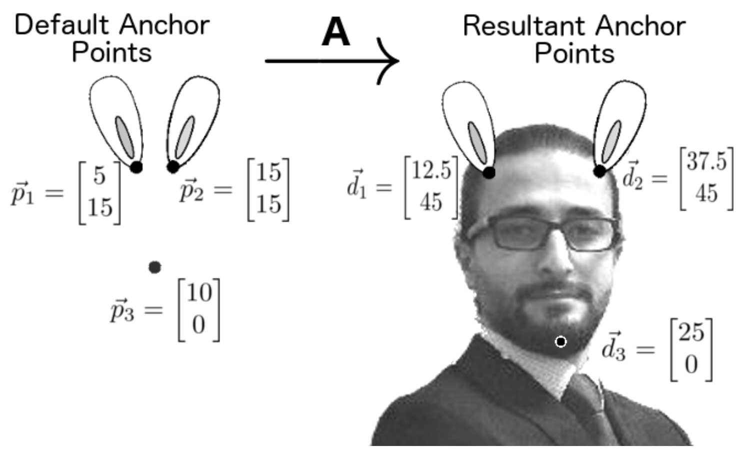

Figure 8.2: Daniel as a bunny.

Solution:

We solve the systems of linear equations

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 5 \\ 15 \end{bmatrix} = \begin{bmatrix} 12.5 \\ 45 \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} 37.5 \\ 45 \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 0 \end{bmatrix}$$

which yield the following equations

$$\begin{cases} 5a = 12.5 \\ 15a = 37.5 \\ 10a = 25 \\ 15b = 45 \\ 15b = 45 \\ 0b = 0 \end{cases}$$

which are solved by $a = 2.5$ and $b = 3$ to finally get:

$$\mathbf{A} = \begin{bmatrix} 2.5 & 0 \\ 0 & 3 \end{bmatrix}$$

(b) (4 points) Rotation is also important to properly align a filter. Say the user's head is tilted 30° **clockwise**, resulting in the anchor points being at $\vec{r}_1, \vec{r}_2$ and $\vec{r}_3$ (see Figure 8.3). **Derive a linear transformation matrix B that appropriately rotates the default anchor points to match the user's head. Precisely, the derived matrix should achieve B$\vec{p}_i = \vec{r}_i$ for all $i \in \{1,2,3\}$. If this is not possible, prove that it is not possible.**

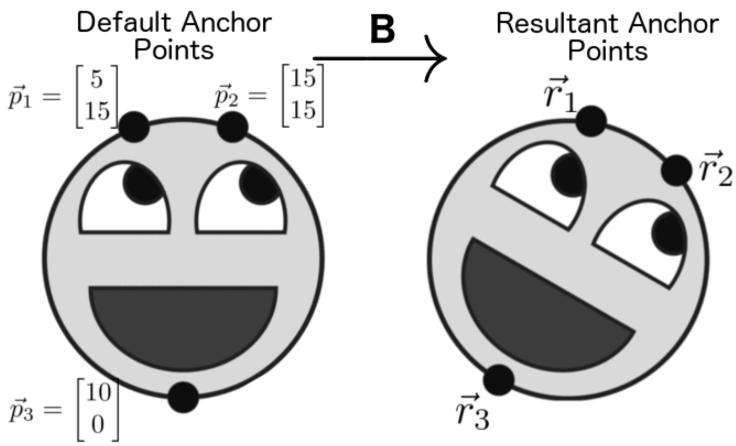

Figure 8.3: Rotation by 30° clockwise.

Solution: This is a rotation by $-30°$ counter-clockwise, which by the rotation formula we learned in class is:

$$\mathbf{B} = \begin{bmatrix} \cos(-30) & -\sin(-30) \\ \sin(-30) & \cos(-30) \end{bmatrix} = \begin{bmatrix} \cos(30) & \sin(30) \\ -\sin(30) & \cos(30) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

(c) (2 points) Filters should also work on reflections! Vasuki has a mirror positioned as shown in Figure 8.4. The mirror results in Vasuki's anchor points to be at $\vec{v}_1 = \begin{bmatrix} -5 \\ 15 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -15 \\ 15 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$. **Derive a linear transformation matrix C that appropriately reflects the default anchor points to match Vasuki's reflection's face. Precisely, the derived transformation should achieve $\mathbf{C}\vec{p}_i = \vec{v}_i$ for all $i \in \{1,2,3\}$. If this is not possible, prove that it is not possible.**

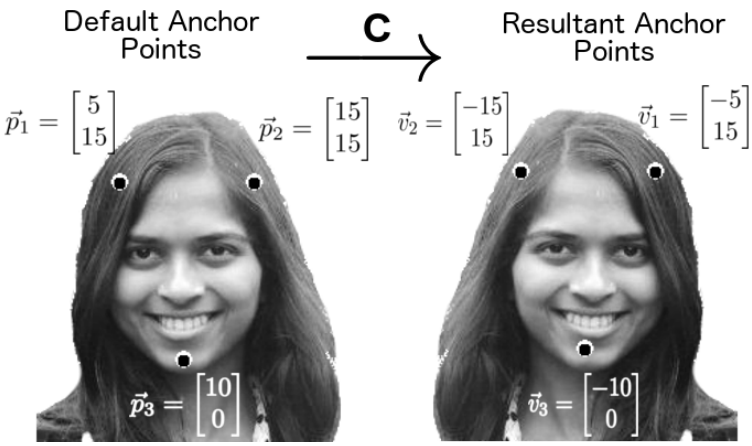

Figure 8.4: Vasuki's face mirrored.

Solution: We solve the systems of linear equations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 5 \\ 15 \end{bmatrix} = \begin{bmatrix} -5 \\ 15 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} -15 \\ 15 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

which yield the following equations

$$\begin{cases} 5a + 15b = -5 \\ 5c + 15d = 15 \\ 15a + 15b = -15 \\ 15c + 15d = 15 \\ 10a = -10 \\ 10c = 0 \end{cases}$$

which are solved by $a = -1, b = 0, c = 0$ and $d = 1$ to finally get:

$$\mathbf{C} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) (2 points) When the user's head that is both rotated 30° clockwise and then reflected in the mirror, the resultant anchor points are $\vec{w}_1, \vec{w}_2$, and $\vec{w}_3$ (see Figure 8.5). We would like to derive a linear transformation matrix **D** that transforms the default anchor points to the user's head in this scenario. Precisely, the derived matrix should achieve $\mathbf{D}\vec{p}_i = \vec{w}_i$ for all $i \in \{1,2,3\}$. **Give an expression for D in terms of B and C (you don't have to explicitly calculate the matrix D). If this is not possible, prove that it is not possible.**

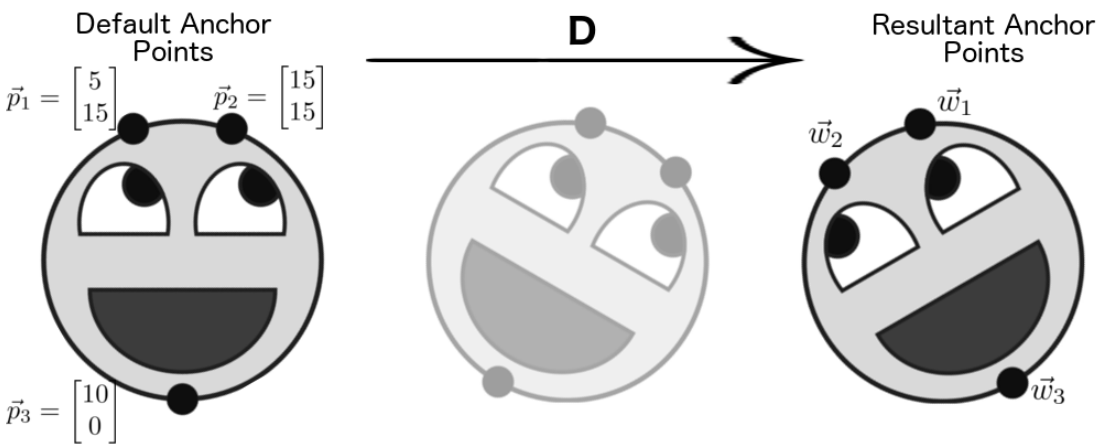

Figure 8.5: Rotation by 30° clockwise followed by mirroring.

Solution:

Order matters! You have to multiply them sequentially from the left.

Thus, the transformation matrix for a rotation by $\angle 30$ clockwise followed by a reflection would be $\mathbf{D} = \mathbf{CB}$.

$$\mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

If you multiply them in the opposite order, you get a reflection first and then a rotation by 30° clockwise!

(e) (4 points) Fil decides that he wants to design his own Snapchat filter that will take just the three default anchor points on a user's face and have it offset the user's body by translation along the x axis by 10 units of length. Fil's desired transformation results in the anchor points being located at $\vec{f}_1 = \begin{bmatrix} 15 \\ 15 \end{bmatrix}, \vec{f}_2 = \begin{bmatrix} 25 \\ 15 \end{bmatrix}$, and $\vec{f}_3 = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$ (see Figure 8.6). **Derive a linear transformation matrix E that transforms the default anchor points to Fil's anchor points after translation. Precisely, the derived matrix should achieve $\mathbf{E}\vec{p}_i = \vec{f}_i$ for all $i \in \{1,2,3\}$. If this is not possible, prove that it is not possible.**

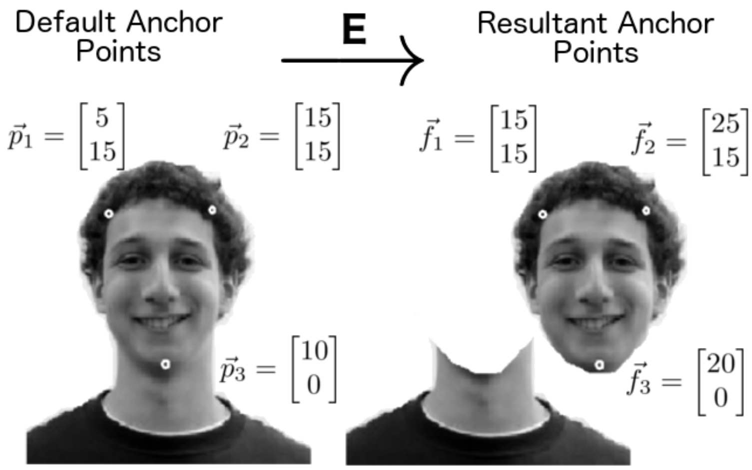

Figure 8.6: The "Fil"-ter.

Solution:

Non-trivial translation is not a linear transformation. To see this, let's assume by contradiction that the requested translation T_E is a linear transformation (equivalent to saying that $\mathbf{E}$ exists: $T_E(\vec{x}) = \mathbf{E}\vec{x}$). Note that $\vec{p}_2 = \vec{p}_1 + \vec{p}_3$. Then we know that since T_E is a linear transformation, we can write $\begin{bmatrix} 25 \\ 15 \end{bmatrix} = \vec{f}_2 \overbrace{=}^{\text{desired}} T_E(\vec{p}_2) \overbrace{=}^{\vec{p}_2=\vec{p}_1+\vec{p}_3} T_E(\vec{p}_1 + \vec{p}_3) \overbrace{=}^{\text{linearity}} T_E(\vec{p}_1) + T_E(\vec{p}_3) \overbrace{=}^{\text{desired}} \vec{f}_1 + \vec{f}_3 = \begin{bmatrix} 35 \\ 15 \end{bmatrix}$. This is a contradiction. Therefore T_E can't be a linear transformation.

The same proof can be written in matrix form as follows. Let's assume by contradiction that such a matrix $\mathbf{E}$ exists. Note that $\vec{p}_2 = \vec{p}_1 + \vec{p}_3$. Therefore, we can write $\begin{bmatrix} 25 \\ 15 \end{bmatrix} = \vec{f}_2 \overbrace{=}^{\text{desired}} \mathbf{E}\vec{p}_2 \overbrace{=}^{\vec{p}_2=\vec{p}_1+\vec{p}_3} \mathbf{E}(\vec{p}_1 + \vec{p}_3) \overbrace{=}^{\text{distributivity}} \mathbf{E}\vec{p}_1 + \mathbf{E}\vec{p}_3 \overbrace{=}^{\text{desired}} \vec{f}_1 + \vec{f}_3 = \begin{bmatrix} 35 \\ 15 \end{bmatrix}$. This is a contradiction. Therefore the matrix $\mathbf{E}$ can't exist.

For fun: there's actually a clever way of linearizing shifts by adding another component to the vectors, which you can read about on Wikipedia. However, the way we phrased the problem is that you can't perform that trick because we demanded that the derived matrix satisfies $\mathbf{E}\vec{p}_i = \vec{f}_i$ for all $i \in \{1,2,3\}$ and $\vec{p}_i$ doesn't have an additional component besides the x and y coordinates.

9. 16A Hogwarts Edition (10 points)

After seeing its success at Berkeley, Hogwarts has decided to add EE16A to its course offerings this semester! You are attending one of their pilot discussion sections on matrix transformations and eigenvalues. In order to teach these concepts, your TA (Hermitian Ranger) has provided a set of cubes. These cubes are matrices that perform a transformation on your position in the coordinate system of your classroom. When the cube is dropped at your feet, it transports you to some new location based on your current one. **Each cube can be modeled with a different matrix transformation T such that $\mathbf{T}\vec{x}_i = \vec{x}_f$ where $\vec{x}_i$ is your initial position and $\vec{x}_f$ is your new position after dropping the cube.**

(a) (2 points) You are given a yellow cube to experiment with. The matrix associated with the yellow cube is **Y**, and your current position in the classroom is $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$. After dropping the yellow cube, you find yourself at position $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$. **Find one of the eigenvalues of Y, λ, and an eigenvector associated with λ. Note that since the yellow cube is given to you, you can't pick an arbitrary Y if you can't uniquely determine Y.**

Solution:

The only acceptable answer is $\lambda_1 = -1$ and any vector $\vec{v}_1 \in \text{span}\left\{\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right\}$ except for $\vec{0}$.

This is straightforward from the problem statement: we know that $\mathbf{Y}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} = (-1)\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ (which is the definition of an eigenvalue/eigenvector pair).

(b) (6 points) You now start from a new location, $\vec{d} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Your TA presents to you a brown cube **B** and two green cubes, $\mathbf{G}_1$ and $\mathbf{G}_2$. She says that you need to use one green cube *and* the brown cube *one after the other*, such that at the end, you are back at your starting point $\vec{d} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Table 9.1 lists the eigenvalue/eigenvector pairs of each of the three cubes in this part. **Which of the two green cubes would you use to achieve this? Prove mathematically that your answer is correct.**

Hint 1: You do not have to explicitly write out the matrices.

Hint 2: In this part, the order in which you use the cubes does not matter.

Table 9.1: Cubes and their eigenpairs.

Green Cubes	Eigenpairs	Brown Cube	Eigenpairs
$\mathbf{G}_1$	$\left(2, \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right), \left(-0.5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$	**B**	$\left(0.5, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right), \left(1, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$
$\mathbf{G}_2$	$\left(1, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right), \left(2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$		

Solution:

Use $\mathbf{G}_2$ and **B**. These would work for any starting location because they are inverses of each other. We can tell that one is the inverse of the other because they share the same eigenvectors and because the corresponding eigenvalues are reciprocals of each other. The proof for this is as follows:

Let's stack the eigenvectors of **B** in a matrix: $\mathbf{V} = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$, then we know that:

$$\mathbf{B} = \mathbf{V}\begin{bmatrix}0.5 & 0\\ 0 & 1\end{bmatrix}\mathbf{V}^{-1}$$

Note that the columns of $\mathbf{V}$ are also the eigenvectors of $\mathbf{G}_2$, so we can write

$$\mathbf{G}_2 = \mathbf{V}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\mathbf{V}^{-1}$$

Let's verify that $\mathbf{G}_2$ and $\mathbf{B}$ are inverses.

$$\mathbf{G}_2\mathbf{B} = \underbrace{\mathbf{V}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\cancel{\mathbf{V}^{-1}}}_{\mathbf{G}_2}\underbrace{\cancel{\mathbf{V}}\begin{bmatrix}0.5 & 0\\ 0 & 1\end{bmatrix}\mathbf{V}^{-1}}_{\mathbf{B}} = \mathbf{V}\overbrace{\cancel{\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0.5 & 0\\ 0 & 1\end{bmatrix}}}^{\mathbf{I}}\mathbf{V}^{-1} = \overbrace{\cancel{\mathbf{V}\mathbf{V}^{-1}}}^{\mathbf{I}} = \mathbf{I}$$

Similarly

$$\mathbf{B}\mathbf{G}_2 = \underbrace{\mathbf{V}\begin{bmatrix}0.5 & 0\\ 0 & 1\end{bmatrix}\cancel{\mathbf{V}^{-1}}}_{\mathbf{B}}\underbrace{\cancel{\mathbf{V}}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\mathbf{V}^{-1}}_{\mathbf{G}_2} = \mathbf{V}\overbrace{\cancel{\begin{bmatrix}0.5 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}}}^{\mathbf{I}}\mathbf{V}^{-1} = \overbrace{\cancel{\mathbf{V}\mathbf{V}^{-1}}}^{\mathbf{I}} = \mathbf{I}$$

Since $\mathbf{B}\mathbf{G}_2 = \mathbf{I}$ and $\mathbf{G}_2\mathbf{B} = \mathbf{I}$, $\mathbf{B}$ must be the inverse of $\mathbf{G}_2$ and vice versa.
This implies: $\mathbf{B}\mathbf{G}_2\vec{d} = \vec{d}$ and we are back where we started as requested.

(c) (2 points) **Would the same choice of green cube from part (b) still work if you started from a different location? Justify your answer.**

Solution:

Since $\mathbf{G}_2$ and $\mathbf{B}$ are inverses of each other, you will always land at wherever you started no matter what.

PRINT your name and student ID: ____

10. Aww, nuts! (18 points)

In all of your spare time outside of EE 16A, you have started working as a server at San Francisco's newest bougie chic boutique restaurant, "Brasserie $E^2 2^4 A$: The Peanut Gallery," featuring peanut-infused dishes from around the world. One hipster feature of the restaurant's kitchen is a "peanut sensor" used to prevent delivering plates with allergens to people with peanut allergies. In the figures, each sensor measurement is represented by an arrow, and the output measurement (labeled at the end of the arrow) is equal to the sum of the individual dish readings. **For illustration purposes only**, Figure 10.1 shows an example of a measurement b that satisfies $p_1 + p_2 + p_3 = b$.

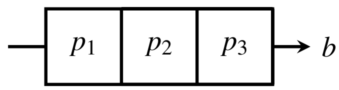

Figure 10.1: Example reading of 3 dishes yielding measurement b **for illustration purposes only**.

Formatting Your Answer: In this problem, whenever you are asked to find all solutions for a vector of variables, say $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$, present the answer in vector format. Examples of acceptable vector formats are:

- $\vec{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.
- $\vec{p} = \begin{bmatrix} 1-3\beta \\ 2-\alpha+2\beta \\ 3+\alpha \end{bmatrix}, \forall \alpha, \beta \in \mathbb{R}$.
- $\vec{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \forall \alpha, \beta \in \mathbb{R}$.

(a) (2 points) In your first order, there are two dishes. You place them in the peanut sensor as depicted in the following figure (Figure 10.2):

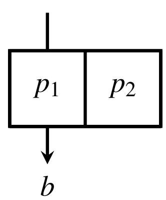

Figure 10.2: Order up! First order.

Let $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ represent the peanut content in both dishes. Given that the result of your peanut measurement is b, **find all possible solutions for $\vec{p}$ in terms of b and any free variables, if needed.**

Solution:

$$\vec{p} = \begin{bmatrix} b \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R}$$

Because the measurement only measures p_1, the value of p_2 can be anything and still fulfill the original b measurement.

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \vec{p} = b$$

This implies that $\vec{p} = \begin{bmatrix} b \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where $\begin{bmatrix} b \\ 0 \end{bmatrix}$ is the particular solution and $\alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the homogeneous solution, giving us $p_1 = b$ and $p_2 = \alpha, \alpha \in \mathbb{R}$.

(b) (2 points) Using the same order from part (a). **Draw the line that represents all possible solutions for $\vec{p}$ for $b = 0$ on the axes below (Figure 10.3) and label the line $b = 0$. Then, on the same set of axes, plot the line that represents all possible solutions for $\vec{p}$ for $b = 1$ and label the line $b = 1$.**

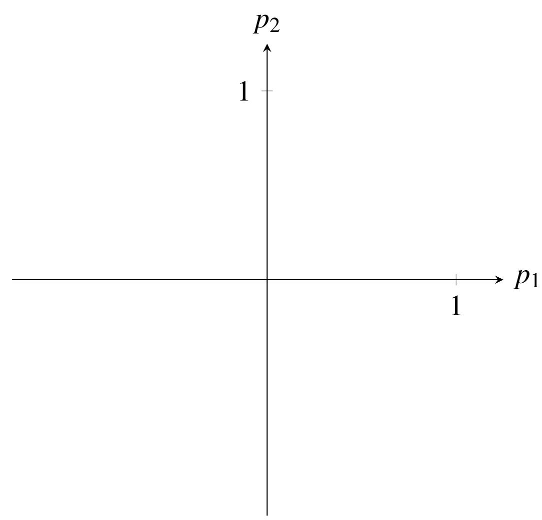

Figure 10.3: Plot the possible peanut content solutions.

Solution:

$$b = 1 : \begin{bmatrix} 1 \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R}$$

$$b = 0 : \begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R}$$

—

We plot those to get:

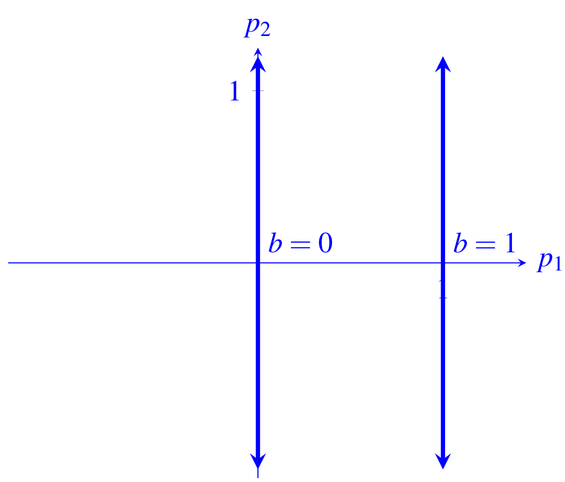

Figure 10.4: Solution to the possible peanut solutions.

(c) This part of the problem consists of two sub-parts (c)i. and (c)ii.. **Only for both of these sub-parts** do we know the following two pieces of extra information. First, the chef now tells you that, for the same order as in parts (a) and (b), one dish contains peanuts, and the other dish has no peanuts. Second, you set the sensor to work as follows: 1) A dish with peanuts will yield a reading of 1; and 2) a dish without peanuts will yield a reading of 0.

i. (4 points) **Now find all possible solutions for** $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ **for both cases when** $b = 0$ **and** $b = 1$**.**

$b = 0$	$b = 1$

Solution:

In essence, the new information in this part adds an extra equation $p_1 + p_2 = 1$.

Our new system of linear equations is now

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 0 & b \end{array}\right]$$

Putting this matrix in reduced row echelon form and solving for $\vec{p}$ would result in a solution.

$$\left[\begin{array}{cc|c} 1 & 0 & b \\ 0 & 1 & 1-b \end{array}\right] \implies \begin{cases} p_1 = b \\ p_2 = 1 - b \end{cases}$$

$$\vec{p} = \begin{bmatrix} b \\ 1-b \end{bmatrix} \implies \begin{cases} \vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b = 1 \\ \vec{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, b = 0 \end{cases}$$

There are only two possible solutions: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $b = 1$ and $b = 0$, respectively.

ii. (1 point) Given the new information presented in this part, is the single measurement $b = 0$ or $b = 1$ now sufficient to uniquely determine which of the two dishes in the first order has peanuts? **Circle your answer.**

YES **NO**

Solution:
Yes. In part (c), we have a unique solution for each of the cases where $b = 0$ and where $b = 1$.

(d) (5 points) Now it's time for your second order. This one has four dishes:

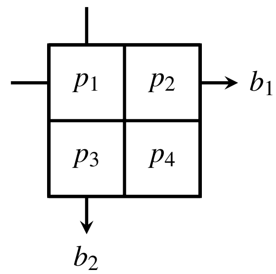

Figure 10.5: Second order!

Let $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$ represent the peanut content in the four dishes. Given the two measurements b_1 and b_2, **find all possible solutions for $\vec{p}$ in terms of b_1, b_2, and any free variables, if needed.**

Solution:

For four dishes, we do the same process we did for two dishes: to find all possible solutions, solve for a particular solution and the homogeneous solution.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & b_1 \\ 1 & 0 & 1 & 0 & b_2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & b_2 \\ 0 & 1 & -1 & 0 & b_1 - b_2 \end{array}\right]$$

We see from the reduced row echelon form that our pivots correspond to p_1 and p_2, meaning our other variables p_3 and p_4 are free variables. For ease, we set $p_3 = p_4 = 0$, resulting in the following equations:

$$\begin{cases} p_1 + p_3 = b_2 \\ p_2 - p_3 = b_1 - b_2 \end{cases} \xrightarrow{p_3=0} \begin{cases} p_1 = b_2 \\ p_2 = b_1 - b_2 \end{cases}$$

This gives us a particular solution $\vec{p}_p$:

$$\vec{p}_p = \begin{bmatrix} b_2 \\ b_1 - b_2 \\ 0 \\ 0 \end{bmatrix}$$

For the general solution, we solve the same augmented matrix for $\vec{b} = 0$.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

Similarly, $p_3 = \alpha$ and $p_4 = \beta$ are free variables.

$$\begin{cases} p_1 + p_3 = 0 \\ p_2 - p_3 = 0 \end{cases} \implies \begin{cases} p_1 + \alpha = 0 \\ p_2 - \alpha = 0 \end{cases} \implies \begin{cases} p_1 = -\alpha \\ p_2 = \alpha \end{cases}$$

and so we get the following general homogeneous solution:

$$\vec{p}_h = \begin{bmatrix} -\alpha \\ \alpha \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \forall \alpha, \beta \in \mathbb{R}$$

Finally, we combine our particular and homogeneous solutions to get a complete solution of our $\vec{p}$ vector:

$$\vec{p} = \vec{p}_p + \vec{p}_h = \begin{bmatrix} b_2 \\ b_1 - b_2 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \forall \alpha, \beta \in \mathbb{R}$$

(e) (4 points) **Only for this part**, the chef now tells you that, for the same order as in part (d), it is one of the four following layouts of peanut dishes with each set of four dishes containing exactly three peanut dishes (see Figure 10.6). Moreover, only for this part, you once again set the sensor to work as follows: 1) A dish with peanuts will yield a reading of 1; and 2) a dish without peanuts will yield a reading of 0.

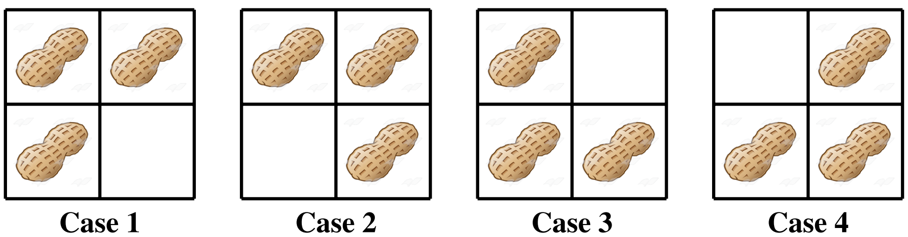

Figure 10.6: Possible Peanut Portioning.

For each of the given arrangements in Figure 10.6, based on the measurement scheme in Figure 10.5, **fill in the values for the measurements b_1 and b_2 in the table below.**

	Case 1	**Case 2**	**Case 3**	**Case 4**
b_1				
b_2				

Solution:

Following the scheme above, the resulting scans for each case are as follows:

	Case 1	**Case 2**	**Case 3**	**Case 4**
b_1	2	2	1	1
b_2	2	1	2	1

11. StateRank Car Rentals (21 points)

You are an analyst at StateRank Car Rentals, which operates in California, Oregon, and Nevada. You are hired to analyze the number of rental cars going into and out of each of the three states (CA, OR, and NV).

The number of cars in each state on day $n \in \{0,1,\ldots\}$ can be represented by the state vector $\vec{s}[n] = \begin{bmatrix} s_{\text{CA}}[n] \\ s_{\text{OR}}[n] \\ s_{\text{NV}}[n] \end{bmatrix}$.

The state vector follows the state evolution equation $\vec{s}[n+1] = \mathbf{A}\vec{s}[n], \forall n \in \{0,1,\ldots\}$, where the transition matrix, $\mathbf{A}$, of this linear dynamic system is

$$\mathbf{A} = \begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix}.$$

(a) (3 points) Use the designated boxes in Figure 11.1 to **fill in the weights for the daily travel dynamics of rental cars between the three states**, as described by the state transition matrix $\mathbf{A}$.
Note the order of the elements in the state vector $\vec{s}[n]$.

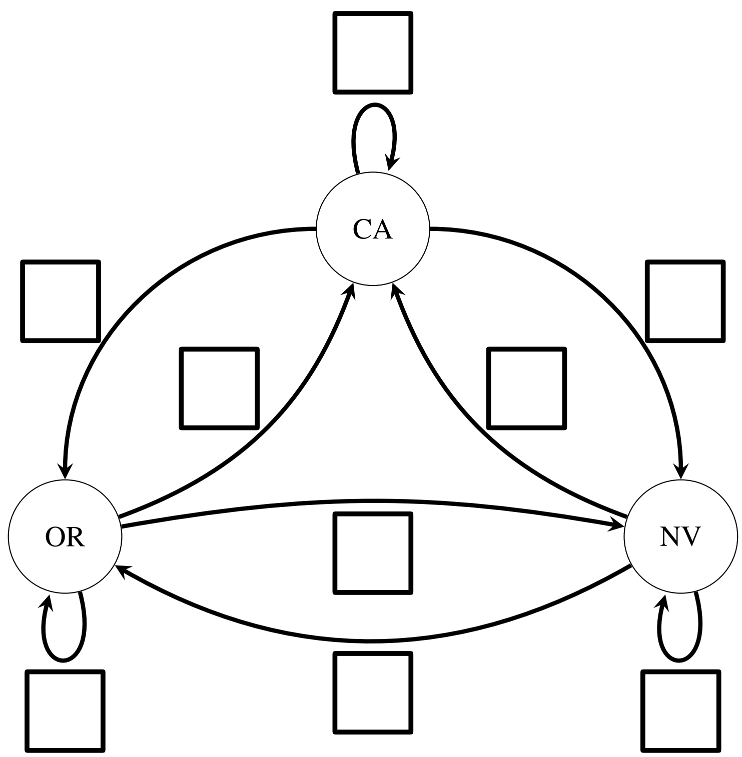

Figure 11.1: StateRank Rental Cars Daily Travel Dynamics.

Solution:

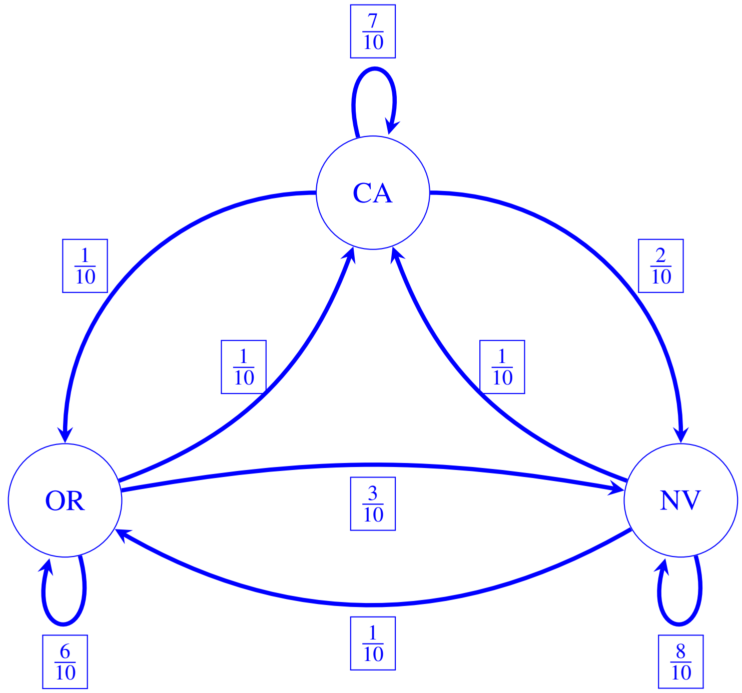

Figure 11.2: StateRank Rental Cars Daily Travel Dynamics – Solution.

A copy of the state transition matrix for reference: $\mathbf{A} = \begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix}$

(b) (2 points) Suppose the state vector on day $n = 4$ is $\vec{s}[4] = \begin{bmatrix} 100 \\ 200 \\ 100 \end{bmatrix}$. **Calculate the state vector on day 5, $\vec{s}[5]$.**

Solution:

$$\begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 70+20+10 \\ 10+120+10 \\ 20+60+80 \end{bmatrix} = \begin{bmatrix} 100 \\ 140 \\ 160 \end{bmatrix}$$

(c) (2 points) We want to express the number of cars in each state on day n as a function of the initial number of cars in each state on day 0. That is, we write $\vec{s}[n]$ in terms of $\vec{s}[0]$ as follows:

$$\vec{s}[n] = \mathbf{B}\vec{s}[0]$$

Express the matrix B in terms of A and n.

Solution:

$$\vec{s}[n] = \mathbf{A}\vec{s}[n-1] = \mathbf{A}^2\vec{s}[n-2] = \mathbf{A}^3\vec{s}[n-3] = \cdots = \mathbf{A}^n\vec{s}[n-n] = \mathbf{A}^n\vec{s}[0]$$

$$\mathbf{B} = \mathbf{A}^n$$

A copy of the state transition matrix for reference: $\mathbf{A} = \begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix}$

(d) (4 points) We denote the eigenvalue/eigenvector pairs of the matrix $\mathbf{A}$ by

$$\left(\lambda_1 = 1, \vec{u}_1 = \begin{bmatrix} 50 \\ 40 \\ 110 \end{bmatrix}\right), \left(\lambda_2, \vec{u}_2 = \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}\right), \text{ and } \left(\lambda_3, \vec{u}_3 = \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix}\right).$$

Find the eigenvalues λ_2 and λ_3 corresponding to the eigenvectors $\vec{u}_2$ and $\vec{u}_3$, respectively. Note that since $\lambda_1 = 1$ is given, you don't have to calculate it.

Solution:

Recall that if $\vec{u}, \lambda$ are an eigenpair of a matrix $\mathbf{A}$, then $\mathbf{A}\vec{u} = \lambda\vec{u}$. By left-multiplying the eigenvectors by $\mathbf{A}$, we get:

i. $\begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix} \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix} = 0.5 \cdot \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}$, which means that $\lambda_2 = 0.5 = \frac{1}{2}$.

ii. $\begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix} \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 6 \end{bmatrix} = 0.6 \cdot \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix}$, which means that $\lambda_3 = 0.6 = \frac{3}{5}$.

(e) (2 points) For the given dynamics in this problem, **does a matrix $\mathbf{C}$ exists such that $\vec{s}[n-1] = \mathbf{C}\vec{s}[n]$, for $n \in \{1, 2, \ldots\}$? Justify your answer.**

Solution:

Yes. The matrix $\mathbf{A}$ is invertible because no eigenvalue is equal to 0.

A copy of the state transition matrix for reference: $\mathbf{A} = \begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix}$

(f) (8 points) Suppose that the initial number of rental cars in each state on day 0 is

$$\vec{s}[0] = \begin{bmatrix} 7000 \\ 5000 \\ 8000 \end{bmatrix} = 100\vec{u}_1 - 100\vec{u}_2 - 200\vec{u}_3,$$

where $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$ are the eigenvectors from part (d).
After a very large number of days n, how many rental cars will there be in each state?
That is, i) calculate

$$\vec{s}^* = \lim_{n\to\infty} \vec{s}[n]$$

and ii) show that the system will indeed converge to $\vec{s}^*$ as $n \to \infty$ if it starts from $\vec{s}[0]$.
Hint: If you didn't solve part (d), the eigenvalues satisfy $\lambda_1 = 1, |\lambda_2| < 1$ and $|\lambda_3| < 1$.

Solution:

We know that $\vec{s}[n] = \mathbf{A}^n\vec{s}[0]$. We also know that $\vec{s}[0] = 100\vec{u}_1 - 100\vec{u}_2 - 200\vec{u}_3$.
Therefore, we can write:

$$\begin{aligned} \vec{s}[n] &= \mathbf{A}^n\vec{s}[0] \\ &= \mathbf{A}^n\left(100\vec{u}_1 - 100\vec{u}_2 - 200\vec{u}_3\right) \\ &= 100\mathbf{A}^n\vec{u}_1 - 100\mathbf{A}^n\vec{u}_2 - 200\mathbf{A}^n\vec{u}_3 \\ &= 100\lambda_1^n\vec{u}_1 - 100\lambda_2^n\vec{u}_2 - 200\lambda_3^n\vec{u}_3 \end{aligned}$$

From that, we can write:

$$\lim_{n\to\infty} \vec{s}[n] = \lim_{n\to\infty} 100\lambda_1^n\vec{u}_1 - 100\lambda_2^n\vec{u}_2 - 200\lambda_3^n\vec{u}_3$$

Since $|\lambda_2| < 1$ and $|\lambda_3| < 1$, we know that:

$$\lim_{n\to\infty} \left(-100\lambda_2^n\vec{u}_2 - 200\lambda_3^n\vec{u}_3\right) = \vec{0}$$

From that and the fact that $\lambda_1 = 1$, we are left with:

$$\begin{aligned} \lim_{n\to\infty} \vec{s}[n] &= \lim_{n\to\infty} 100\lambda_1^n\vec{u}_1 \\ &= \lim_{n\to\infty} 100 \cdot 1^n\vec{u}_1 \\ &= 100\vec{u}_1 \\ &= \begin{bmatrix} 5000 \\ 4000 \\ 11000 \end{bmatrix} \end{aligned}$$

Extra page for scratchwork.

If you want any work on this page to be graded, please refer to this page on the problem's main page.