EECS 16A Designing Information Devices and Systems I
Summer 2022 Homework 1

This homework is due Friday, July 1st, 2022, at 23:59.
Self grades are due Monday, July 4th, 2022, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

1. Reading Assignment [UNGRADED]

For this homework, please read Note 0, Note 1A, Note 1B, Note 2A, Note 2B, and Note 3 (Only Span). These will provide an overview of linear equations, augmented matrices, Gaussian elimination, vectors, matrices, matrix/vector operations, and span. You are always welcome and encouraged to read ahead as well. How does the content you read in these notes relate to what you've learned before? What content is unfamiliar or new?

2. Counting Solutions

Learning Goal: *(This problem is designed to illustrate the different types of systems of equations. Some sub-parts will have a unique solution and others have no solutions or infinitely many solutions. In this class, we will build up the mathematical machinery to systematically determine which case applies.)*

Directions:

For each of the following systems of linear equations, determine if there is a unique solution, no solution, or an infinite number ofsolutions.

If there is a unique solution, find it.

If there is an infinite number of solutions, explicitly state this, describe the set of all solutions, and then write one of such solutions. You can represent the set of all solutions by writing two variables as a function of the last variable (i.e. $x = 2z+1, y = 4z-4$).

If there is no solution, explain why. **Show your work**.

Example: The below example shows how to methodically solve systems of linear equations using the substitution method.

$$\begin{aligned} 2x &+ 3y &= 5 \\ x &+ y &= 2 \end{aligned}$$

Example Solution

$$2x+3y = 5 \tag{1}$$
$$x+y = 2 \tag{2}$$

Subtract: Eq (1) - 2*Eq (2)

$$y = 1 \tag{3}$$

Now we plug in Eq (3) into Eq (2) and solve for x

$$\begin{aligned} x + \ 1 &= 2 \\ \rightarrow \quad x &= 1 \end{aligned} \tag{4}$$

From Eq (3) and Eq (4), we get the unique solution:

$$\begin{aligned} x &= 1 \\ y &= 1 \end{aligned}$$

(a)

$$\begin{array}{rcrcrl} x & + & y & + & z & = 3 \\ 2x & + & 2y & + & 2z & = 5 \end{array}$$

(b)

$$\begin{array}{rcrcrl} & - & y & + & 2z & = 1 \\ 2x & & & + & z & = 2 \end{array}$$

(c)

$$\begin{array}{rcrl} x & + & 2y & = 5 \\ 2x & - & y & = 0 \\ 3x & + & y & = 5 \end{array}$$

(d)

$$\begin{array}{rcrl} x & + & 2y & = 3 \\ 2x & - & y & = 1 \\ x & - & 3y & = -5 \end{array}$$

3. Filtering Out The Troll

You attended a very important public speech and recorded it using a recording device that consists of two directional microphones. However, there was this particular person in the audience who was trolling around. When you went back home to listen to the recording, you realized that the two recordings were dominated by the troll and you could not hear the speech. Fortunately, since you had two microphones, you realized that there is a way to combine the two recordings such that the trolling is removed. Recollecting the scene, the locations of the speaker and the troll are shown in Figure 1.

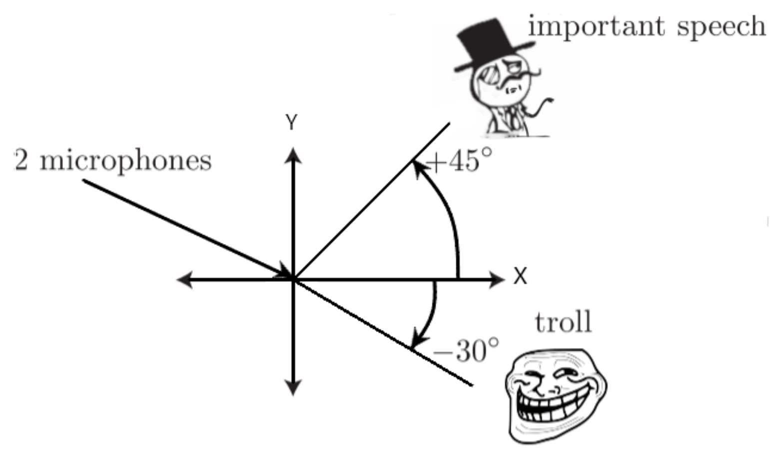


Figure 1: Locations of the speaker and the troll.

The way your recording device works is that each microphone weighs the audio signal depending on the angle of the audio source, relative to the x axis, hence the name *directional microphones*. More specifically, if the audio source is located at an angle of θ, the first microphone will record the audio signal with weight $f_1(\theta) = \cos(\theta)$, and the second microphone will record the audio signal with weight $f_2(\theta) = \sin(\theta)$. For example, an audio source that lies on the x axis will be recorded with the first microphone with weight equal to 1 (since $\cos(0) = 1$), but will not be picked up by the second microphone (since $\sin(0) = 0$). Note that the weights can also be negative.

Graphically, the directional characteristics of the microphones are given in Figures 2 and 3 (the red and blue colors denote the positive and negative values of the weight, respectively and the distance of the red or the blue line from the midpoint is the value of the weight). Putting all of this together, assume that there are two speakers, A and B, at angles θ and ψ, respectively. Assume that speaker A produces an audio signal represented by the vector $\vec{a} \in \mathbb{R}^n$. That is, the i-th component of $\vec{a}$ is the signal at the i-th time step. Similarly, assume speaker B produces an audio signal $\vec{b}$,

Then the first microphone will record the signal

$$\vec{m}_1 = \cos(\theta) \cdot \vec{a} + \cos(\psi) \cdot \vec{b},$$

and the second microphone will record the signal

$$\vec{m}_2 = \sin(\theta) \cdot \vec{a} + \sin(\psi) \cdot \vec{b}.$$

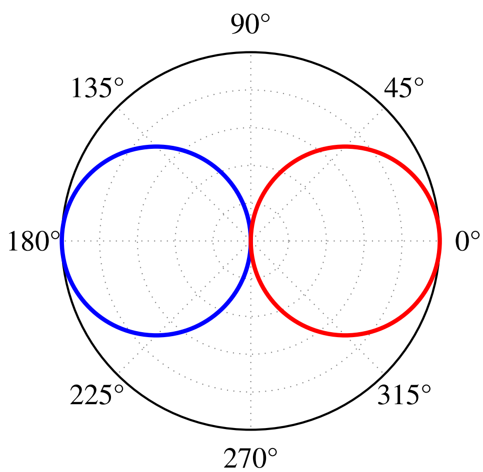


Figure 2: Directional characteristics of mic. 1. The red and blue colors denote the positive and negative values of the weight, respectively. The distance of the red or the blue line from the midpoint is the value of the weight.

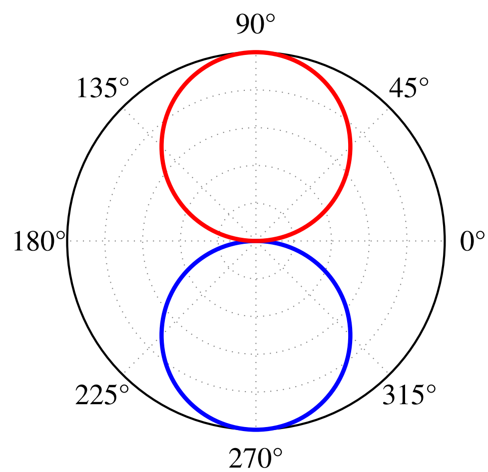


Figure 3: Directional characteristics of mic. 2. The red and blue colors denote the positive and negative values of the weight, respectively. The distance of the red or the blue line from the midpoint is the value of the weight.

(a) Using the notation above, let the important speaker be speaker A (with signal $\vec{a}$) and let the person trolling be speaker B (with signal $\vec{b}$). Express the recordings of the two microphones $\vec{m}_1$ and $\vec{m}_2$ (i.e. the signals recorded by the first and the second microphones, respectively) as a linear combination of $\vec{a}$ and $\vec{b}$.

(b) Recover the important speech $\vec{a}$, as a weighted combination of $\vec{m}_1$ and $\vec{m}_2$. In other words, write $\vec{a} = u \cdot \vec{m}_1 + v \cdot \vec{m}_2$ (where u and v are scalars). What are the values of u and v?

(c) Partial IPython code can be found in `prob1.ipynb`. Complete the code to get a clean signal of the important speech. What does the speaker say? (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sounded remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sounded almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EE16A.

4. Magic Square

In an $n \times n$ "magic square," all of the sums across each of the n rows, n columns, and 2 diagonals equal magic constant k. For example, in the below magic square, each row, column, and diagonal sums to 34.

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

The magic square is a classic math puzzle, and some of you may have solved these as children by guessing. However, it turns out they can be solved systematically by setting up a system of linear equations!

(a) How many linear equations can you write for an $n \times n$ magic square?

(b) For the generalized magic square below, write out a system of linear equations.
Hint: Set the sum of entries in each row, column, and diagonal equal to k.

x_{11}	x_{12}	x_{13}
x_{21}	x_{22}	x_{23}
x_{31}	x_{32}	x_{33}

(c) Now consider the following square, with some entries filled in. Substitute the known entries into the linear equations you wrote in part (b) to solve for the missing entries x_{11}, x_{12}, x_{32}. Please show the equations you use to solve; credit will not be given for solving by inspection.

x_{11}	x_{12}	8
9	5	1
2	x_{32}	6

5. Image Masks

Learning Objective: *Learn to setup imaging problems with matrices.*

For these word problems, you only need to setup the problem with Gaussian elimination or matrix-vector notation. Of course, you may solve for practice, but no additional credit is awarded.

After your first EECS16A lecture, you decide to try to build a single-pixel camera. You want to take a 2x2 image, i.e. 4 tiles, and based on the first lecture, you choose to take 4 measurements. Recall that each measurement is the sum of the illuminated tiles. For each measurement, you will use a different mask.

(a) Initially, you want to illuminate only one tile for each measurement. That is, you will first illuminate x_1, then you will illuminate x_2, etc. The outputs of your 4 measurements are y_1, y_2, y_3, and y_4 respectively. The 4 measurements you take are shown in Figure 4. Explicitly setup the matrix problem for this in the $A\vec{x} = \vec{b}$ form.

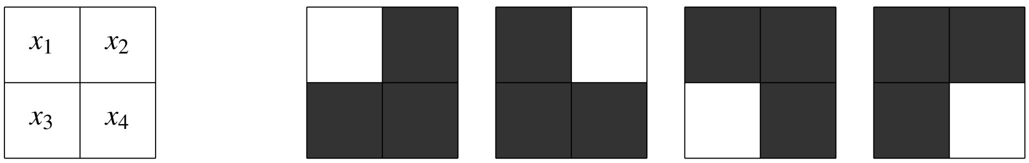
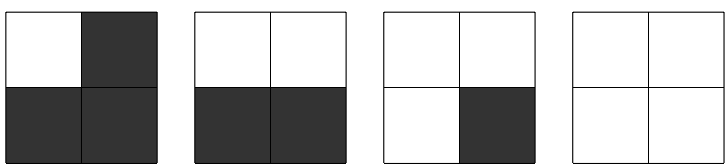


Image tile labels   Measurement 1   Measurement 2   Measurement 3   Measurement 4

Figure 4: Four image masks.

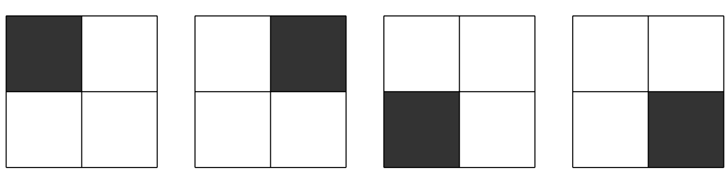
(b) While setting up your code to create the masks, you forget to turn off the illuminated tiles from the previous measurement. As a result, measurement one contains x_1, measurement two contains $x_1 + x_2$, etc. The outputs of your 4 measurements are z_1, z_2, z_3, and z_4 respectively. The 4 measurements you take are shown in Figure 5. Explicitly setup the matrix problem for this in $A\vec{x} = \vec{b}$ form.



Measurement 1   Measurement 2   Measurement 3   Measurement 4

Figure 5: Four image masks.

(c) Your friend is also building their own single pixel camera. However, they make a different mistake in their code and during each measurement, instead of lighting up one tile, you light up the other 3 tiles instead. That is, instead of measuring x_1, they measure $x_2 + x_3 + x_4$. The output of the 4 measurements are w_1, w_2, w_3, and w_4. The 4 measurements from their setup are shown in Figure 6. Explicitly setup the matrix problem for this in $A\vec{x} = \vec{b}$ form.



Measurement 1   Measurement 2   Measurement 3   Measurement 4

Figure 6: Four image masks.

6. Word Problems

Learning Objective: *Understand how to setup a system of linear equations from word problems.*

For these word problems, represent the system of linear equations as an augmented matrix. Then, solve the system using substutition or Gaussian elimination.

(a) Gustav is collecting soil samples. Each soil sample contains some sand, some clay, and some organic material. He wants to know the density of each material. His first sample has 0.5 liters of sand, 0.25 liters of clay, and 0.25 liters of organic material, and weighs 1.625 kg. His second sample contains 1 liter of sand, 0 liters of clay, and 1 liter of organic material, and weighs 3 kg. His third sample contains 0.25 liters of sand, 0.5 liters of clay, and 0 liters of organic material, and weighs 1.375 kg. That is,

$$0.5s + 0.25c + 0.25m = 1.625 \tag{5}$$
$$1s + 0c + 1m = 3 \tag{6}$$
$$0.25s + 0.5c + 0m = 1.375 \tag{7}$$

where s is the density of sand, c is the density of clay, and m is the density of organic material, all measured in kg/L. Solve for the density of each material.

(b) Alice buys 3 apples and 4 oranges for 17 dollars. Bob buys 1 apple and 10 oranges for 23 dollars (Bob really likes oranges). How much do apples and oranges cost individually?

(c) Jack, Jill, and James are driving from Berkeley to Las Vegas. Each of them takes a different route. Jack takes a short route and ends up going through Toll Road A and Toll Road B, costing him $10. Jill takes a slightly longer route and goes through Toll Road B and Toll Road C, costing her $15. Finally, James takes a wrong turn and takes Toll Road A twice, then takes Toll Road B and finally Toll Road C, costing him $25. What is the toll cost on each road?

7. Linearity

In this question, we will explore in further detail what exactly it means for a function to be linear. For each of the following, please specify the values of a (a real number) for which the function is linear. Here, x and y are variables.

(a)
$$f(x,y) = (3-a)x + 2ay$$

(b)
$$f(x,y) = a^2x + 8y$$

(c)
$$f(x,y) = y + axy - 3x$$

(d)
$$f(x,y) = (x+ay)^2$$

8. Gaussian Elimination

Learning Goal: *Understand the relationship between Gaussian elimination and the graphical representation of linear equations, and explore different types of solutions to a system of equations. You will also practice determining the parametric solutions when there are infinitely many solutions.*

(a) In this problem we will investigate the relationship between Gaussian elimination and the geometric interpretation of linear equations. You are welcome to draw plots by hand or using software. Please be sure to label your equations with a legend on the plot.

i. Plot the following set of linear equations in the x-y plane. If the lines intersect, write down the point or points of intersection.

$$x + 2y = 4 \tag{1}$$
$$2x - 4y = 4 \tag{2}$$
$$3x - 2y = 8 \tag{3}$$

ii. Write the above set of linear equations in augmented matrix form and do the first step of Gaussian elimination to eliminate the x variable from equation 2. Now, the second row of the augmented matrix has changed. Plot the corresponding new equation created in this step on the same graph as above. What do you notice about the new line you draw?

iii. Complete all of the steps of Gaussian elimination including back substitution. Now plot the new equations represented by the rows of the augmented matrix in the last step (after completing back substitution) on the same graph as above. What do you notice about the new line you draw?

(b) Write the following set of linear equations in augmented matrix form and use Gaussian elimination to determine if there are no solutions, infinite solutions, or a unique solution. If any solutions exist, determine what they are. You may do this problem by hand or use a computer. We encourage you to try it by hand to ensure you understand Gaussian elimination. Remember that it is possible to end up with fractions during Gaussian elimination.

$$\begin{aligned} x + 2y + 5z &= 3 \\ x + 12y + 6z &= 1 \\ 2y + z &= 4 \\ 3x + 16y + 16z &= 7 \end{aligned}$$

(c) Consider the following system:

$$\begin{aligned} 4x + 4y + 4z + w + v &= 1 \\ x + y + 2z + 4w + v &= 2 \\ 5x + 5y + 5z + w + v &= 0 \end{aligned}$$

If you were to write the above equations in augmented matrix form and use Gaussian elimination to solve the system, you would get the following (for extra practice, you can try and do this yourself):

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 16 \\ 0 & 0 & 1 & 0 & -3 & -17 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

How many variables are free variables? Determine the solutions to the set of equations.

9. Vector-Vector Multiplication

Learning Objective: *Practice evaluating vector-vector multiplication.*

For the following multiplications, state the dimensions of the result. If the product is not defined and thus has no solution, state this and justify your reasoning. For this problem $\vec{x} \in \mathbb{R}^N, \vec{y} \in \mathbb{R}^N, \vec{z} \in \mathbb{R}^M$, with $N \neq M$.

(a) i. $\vec{x}^T \cdot \vec{z}$

ii. $\vec{x} \cdot \vec{x}^T$

iii. $\vec{x} \cdot \vec{y}^T$

iv. $\vec{x} \cdot \vec{z}^T$

10. Multiply the Matrices

Learning Objective: *Practice evaluating matrix-matrix multiplication.*

(a) We have two matrices **A** and **B**, where **A** is a 3×2 matrix and **B** is a 2×4 matrix. Would the multiplication **AB** be a valid operation? If yes, what do you expect the dimensions of **AB** to be?

(b) Compute **AB** by hand, where **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & 0 & 2 & -1 \end{bmatrix}$$

Compute **BA** too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

(c) Compute **AB** by hand, where **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 3 & 21 & 9 \\ -1 & 14 & 4 \\ 7 & -8 & 2 \end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Compute **BA** too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

11. Vectors in the Span

Learning Goal: *Practice determining whether a vector is in the span of a set of vectors.*

Determine whether a vector $\vec{v}$ is in the span of the given set of vectors. If it is in the span of given set, write $\vec{v}$ as a linear combination of given set of vectors (you will need to find the scalar coefficients in the linear combination).

(a) $\vec{v} = \begin{bmatrix} -10 \\ 4 \end{bmatrix}$ and $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

(b) $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ and $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix} \right\}$

(c) $\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \right\}$

(d) $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} \right\}$

12. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Please remember to submit both your homework as well as the self-grade assignment following the release of the solutions. A full description of the submission process is listed on the class website (eecs16a.org).