Lecture 2B (07/05/22)

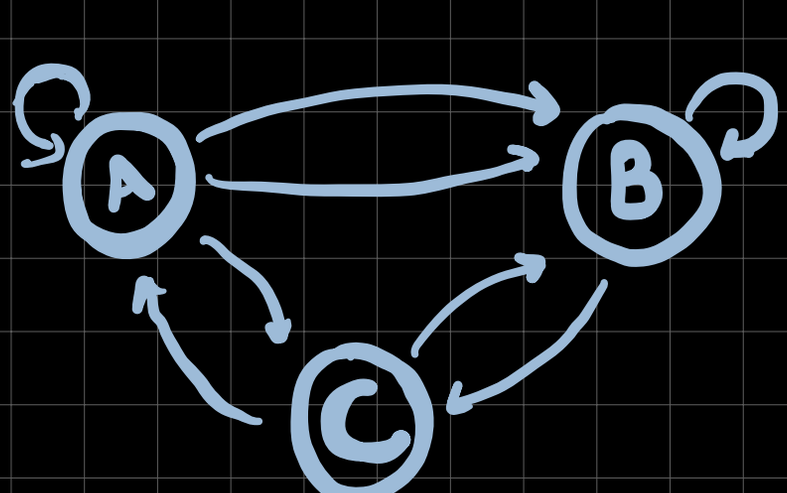
Announcements

- Quest is next Monday! Logistics post will be up on Ed.
  - → 2:10 → 3:10pm in Li Ka Shing 245

Agenda

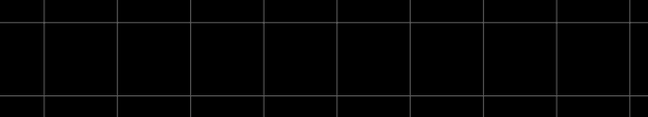
- State Transition Systems
- Matrix Inverses
- Vectorspaces

Setup: Review



$$\vec{x}[t] = \begin{bmatrix} q_A[t] \\ q_B[t] \\ q_C[t] \end{bmatrix}$$

$$\vec{x}[t+1] = M\vec{x}[t]$$



"Discrete Time"

Setup

$$M = \begin{bmatrix} p_{AA} & p_{BA} & p_{CA} \\ p_{AB} & p_{BB} & p_{CB} \\ p_{AC} & p_{BC} & p_{CC} \end{bmatrix}$$

p_{AB} = proportion moving from A to B

$$\begin{bmatrix} q_A[t+1] \\ q_B[t+1] \\ q_C[t+1] \end{bmatrix} = \begin{bmatrix} p_{AA} q_A[t] + p_{BA} q_B[t] + p_{CA} q_C[t] \\ p_{AB} q_A[t] + p_{BB} q_B[t] + p_{CB} q_C[t] \\ p_{AC} q_A[t] + p_{BC} q_B[t] + p_{CC} q_C[t] \end{bmatrix}$$

Setup

$$x[t+1] = Mx[t]$$

$$\downarrow$$

$$x[t+2] = Mx[t+1]$$

$$\downarrow$$

$$x[t+2] = M(Mx[t])$$

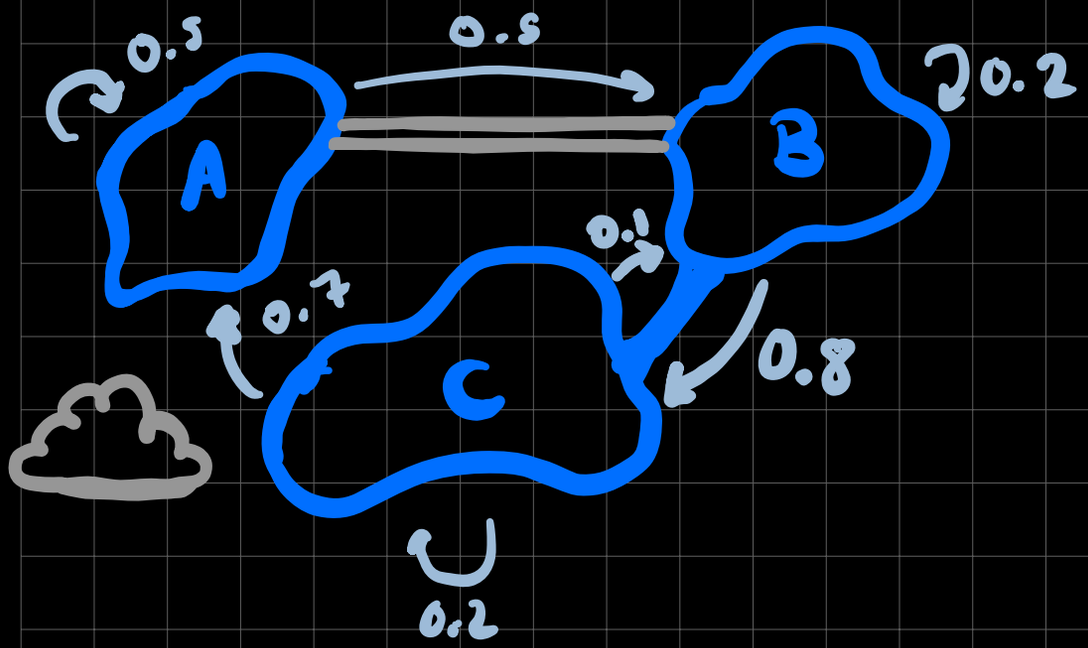
$$\downarrow$$

$$x[t+2] = M^2 x[t]$$

$$\star \quad x[t] = M^t x[0]$$

Consider $M = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$, what happens as $t \to \infty$?

Water Reservoirs: Example



If $\vec{x}[0] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

What is $x[1]$?

$$M = \begin{bmatrix} p_{AA} & p_{BA} & p_{CA} \\ p_{AB} & p_{BB} & p_{CB} \\ p_{AC} & p_{BC} & p_{CC} \end{bmatrix}$$

p_{AB} = proportion moving from A to B

Conservation Property

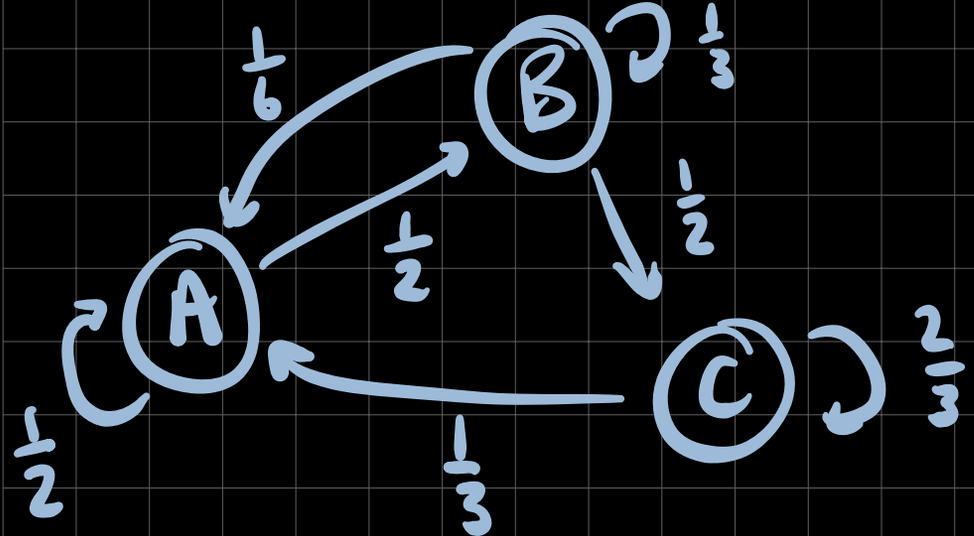
A system is conservative iff the columns of the transition matrix sum to 1

Are the following conservative?

$$\begin{bmatrix} 0.2 & 0.3 & 1 \\ 0.8 & 0.1 & 0 \\ 0 & 0.6 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.2 & 0.8 \\ 1 & 0 & 0 \end{bmatrix}$$

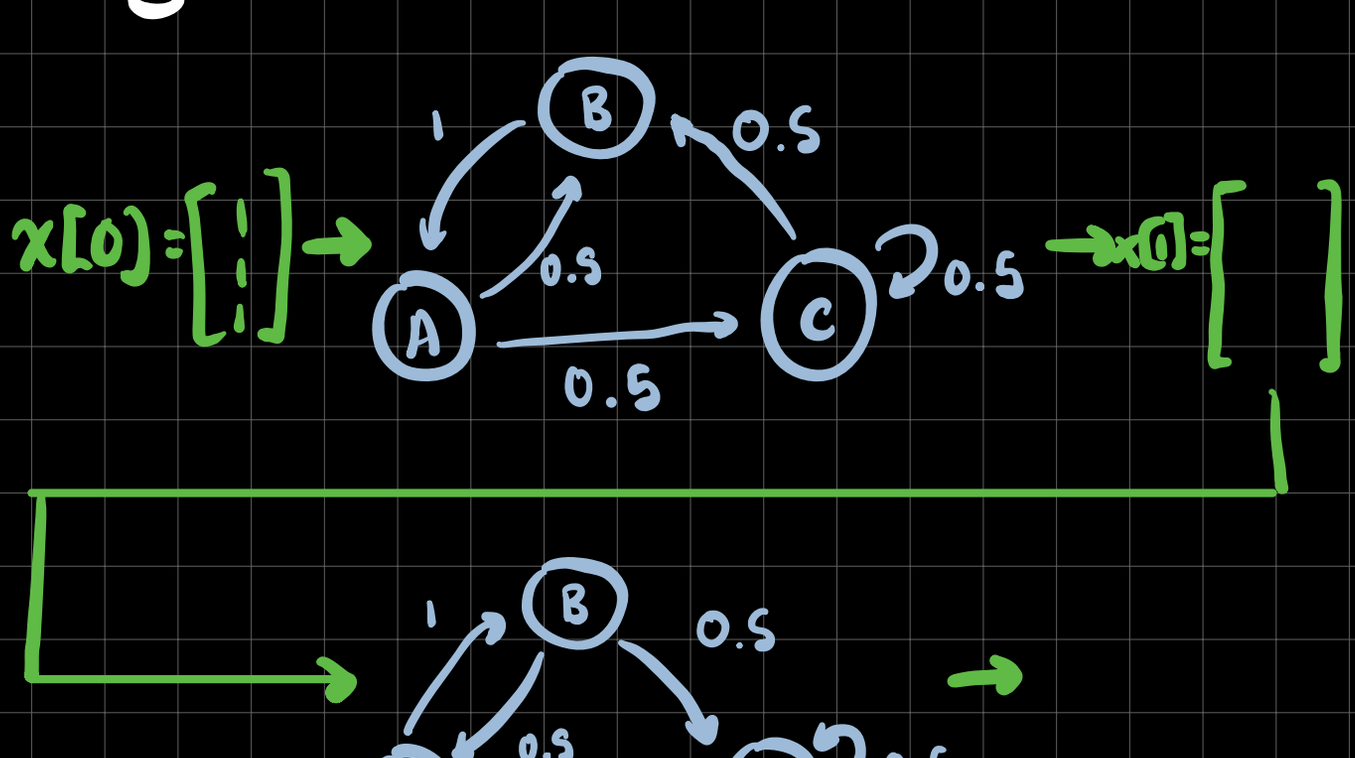
Intuitive Meaning:

Water Reservoir: Example 2



Write out the transition matrix M. Is it conservative?

Moving Back in Time?



Moving Back in Time?

Given M and $\vec{x}[t+1]$, can we find $\vec{x}[t]$?

(i.e. can we find B s.t. $B\vec{x}[t+1] = \vec{x}[t]$)

- Scalar Case:

$$x[t+1] = mx[t]$$

Matrix Inversion

$$Mx[t] = x[t+1]$$

$$\Downarrow$$

$$x[t] = M^{-1}x[t+1]$$

Inverse

Def. A square matrix M is invertible if $\exists M^{-1}$ such that

$$M(M^{-1}) = (M^{-1})M = I$$

Uniqueness of Inverse

Thm. The inverse of a square matrix is unique

Proof:

Finding the Inverse

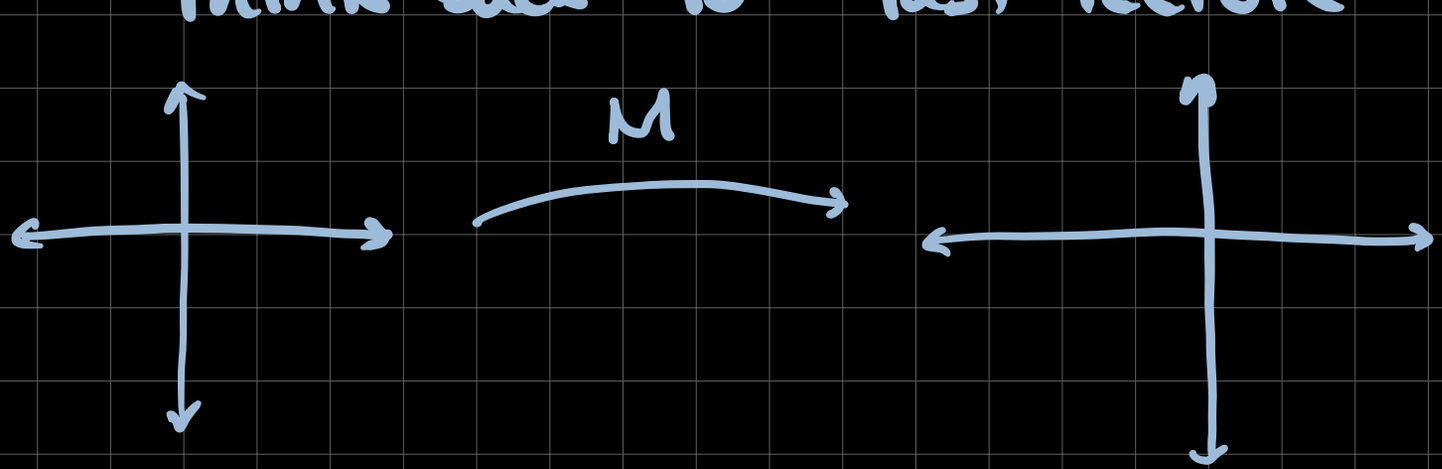
$$\underbrace{M}_{\text{known}} M^{-1} = \underbrace{I}_{\text{known}}$$ ⇐ How do we solve?

Example

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

Does an inverse always exist?

Think back to last lecture



Inverse: Equivalent Statements

Let A be a square matrix, the following statements are equivalent

1. A is invertible
2. A has linearly independent columns
3. A has linearly independent rows
4. $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$
5. A has a trivial null space
6. The determinant of $A \neq 0$

} next lectures

Proof $\exists A^{-1} \Leftrightarrow$ lin. ind. columns

Inverse of 2×2: Shortcut

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$ad-bc$ is the determinant of A

Vectorspaces

- **Def:** Set of vectors and scalars (V, F) and two operators $\cdot, +$ that satisfy the following:

1. $\alpha\vec{x} \in V$
2. $\vec{x}+\vec{y} \in V$
3. $\vec{x}+(\vec{y}+\vec{z}) = (\vec{x}+\vec{y})+\vec{z}$ (associativity)
4. $\vec{x}+\vec{y} = \vec{y}+\vec{x}$ (commutativity)
5. $\exists\, \vec{0}$ s.t. $\vec{x}+\vec{0} = \vec{x}$ (additive identity)
6. $\exists (-\vec{x})$ s.t. $\vec{x}+(-\vec{x}) = 0$ (additive identity)
7. $\alpha(\vec{x}+\vec{y}) = \alpha\vec{x}+\alpha\vec{y}$ (distributivity)
8. $\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$ (multiplicative associativity)
9. $\exists 1$ s.t. $1\vec{x} = \vec{x}$
10. $(\alpha+\beta)\vec{x} = \alpha\vec{x}+\beta\vec{x}$

Vector Spaces: Examples