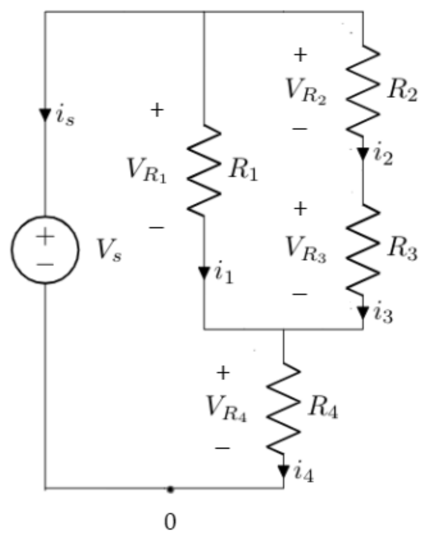
EECS 16A Designing Information Devices and Systems I
Fall 2021 Midterm 1

1. Circuits Part I (6 points)

You are given the following circuit for Question 1-2.



(a) What is the number of nodes in the circuit, including the ground node?

(b) Complete the following KCL equations:

i_s + [value1] + [value2] = 0

[value3] - i_3 + [value4] = 0

(c) Complete the following Ohm's law relationships:
$V_{R2} = i_2[value5]$

2. Circuits Part II (3 points)

(d) When solving the circuit using $\mathbf{A}\vec{x} = \vec{b}$, $\vec{x}$ consists of all element currents and node potentials, other than the zero potential node. How many elements are there in $\vec{x}$?

(e) How many columns are there in $\mathbf{A}$?

3. Gaussian Elimination Part I (3 points)

(version 1) Complete the following Gaussian elimination problem by filling the blanks for the equation $\mathbf{A}\vec{x} = \vec{b}$, where $\mathbf{A} = \begin{bmatrix} 2 & 6m \\ 2 & 12 \\ 2 & 12 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 8 \\ k \end{bmatrix}$.

$$\left(\begin{array}{cc|c} 2 & 6m & 2 \\ 2 & 12 & 8 \\ 2 & 12 & k \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & [\text{value1}] & 1 \\ 1 & 6 & 4 \\ 0 & 0 & k+[\text{value2}] \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & [\text{value1}] & 1 \\ 0 & [\text{value3}] & 3 \\ 0 & 0 & k+[\text{value2}] \end{array}\right)$$

(version 2) Complete the following Gaussian elimination problem by filling the blanks for the equation $\mathbf{A}\vec{x} = \vec{b}$, where $\mathbf{A} = \begin{bmatrix} 2 & 8m \\ 2 & 12 \\ 2 & 12 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 8 \\ k \end{bmatrix}$.

$$\left(\begin{array}{cc|c} 2 & 8m & 2 \\ 2 & 12 & 8 \\ 2 & 12 & k \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & [\text{value1}] & 1 \\ 1 & 6 & 4 \\ 0 & 0 & k+[\text{value2}] \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & [\text{value1}] & 1 \\ 0 & [\text{value3}] & 3 \\ 0 & 0 & k+[\text{value2}] \end{array}\right)$$

(version 3) Complete the following Gaussian elimination problem by filling the blanks for the equation $\mathbf{A}\vec{x} = \vec{b}$, where $\mathbf{A} = \begin{bmatrix} 2 & 6m \\ 2 & 12 \\ 2 & 12 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 6 \\ k \end{bmatrix}$.

$$\left(\begin{array}{cc|c} 2 & 6m & 2 \\ 2 & 12 & 6 \\ 2 & 12 & k \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & [\text{value1}] & 1 \\ 1 & 6 & 3 \\ 0 & 0 & k+[\text{value2}] \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & [\text{value1}] & 1 \\ 0 & [\text{value3}] & 2 \\ 0 & 0 & k+[\text{value2}] \end{array}\right)$$

Note that multiple row reduction steps were taken between the arrows.
Express your answers in terms of m and/or k.

value1 = [ ]

value2 = [ ]

value3 = [ ]

4. Gaussian Elimination Part II (6 points)

You performed some row reduction steps to solve the equation $\mathbf{A}\vec{x} = \vec{b}$ and you arrived at the following result:

(version 1) $\left(\begin{array}{cc|c} 1 & 5m & 1 \\ 0 & 2-m & 3 \\ 0 & 0 & 3k \end{array}\right)$

(version 2) $\left(\begin{array}{cc|c} 1 & 5m & 1 \\ 0 & 3-m & 3 \\ 0 & 0 & k-3 \end{array}\right)$

(version 3) $\left(\begin{array}{cc|c} 1 & 5m & 1 \\ 0 & 2-2m & 3 \\ 0 & 0 & 5k \end{array}\right)$

(a) For what value of k could the solution exist?

(b) For what value of m does the system always have no solutions, independent of k?

(c) For what value of m does the matrix **A** have a non-trivial null space?

5. Berkeley boba stores Part I (6 points) *The following description applies to Question 5-6.*
Your EECS 16A TA Francis is a bobaholic. She is hosting a boba party with her lab ASEs to try out a secret recipe with a unique combination of toppings. Instead of buying each topping separately, Francis orders from Berkeley boba stores (U-cha, Yi Fang, Hay Tea) to get the combination of toppings.

Number of drinks	Boba store	Toppings		
		Grass jelly serving	Red bean serving	Mango pudding serving
x_u	U-cha	2	2	2
x_y	Yi Fang	1	3	4
x_h	Hay Tea	4	3	1

The secret recipe	
Toppings	Servings
Grass jelly	10
Red bean	9
Mango pudding	14

$\vec{x} = \begin{bmatrix} x_u \\ x_y \\ x_h \end{bmatrix}$ represents the number of drinks Francis is ordering from each store. $\vec{y}$ represents the servings of each topping Francis will get.

To try out the secret recipe, Francis sets up a system of linear equations $\mathbf{A}\vec{x} = \vec{y}$ to figure out how many drinks she needs from each store.

Fill in the blanks. You do not need to solve for $\vec{x}$

$$A = \begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}, \vec{y} = \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix}$$

6. Berkeley boba stores Part II (7 points)

From another recipe, Francis sets up another system of linear equations: $\mathbf{A}\vec{x} = \vec{y}$ where $\vec{y} \neq \vec{0}$. The solution has this form $\vec{x} = \begin{bmatrix} x_u \\ x_y \\ x_h \end{bmatrix} = \begin{bmatrix} \alpha + 2 \\ -3\alpha + 4 \\ \alpha \end{bmatrix}$, where α is a free variable. Given that we can only add drinks, the number of drinks cannot have a negative value ($x_u \geq 0, x_y \geq 0, x_h \geq 0$). Which of the following statements are implied? (Select all that apply)

(a) rank($\mathbf{A}$) = 1

(b) rank($\mathbf{A}$) = 2

(c) the set of possible solutions forms a subspace

(d) the span of vector $\vec{x}$ forms the null space of $\mathbf{A}$

(e) the set of possible solutions is a subset of a vector space

(f) the set of possible solutions forms an eigenspace

(g) matrix $\mathbf{A}$ has a non-trivial null space

7. Linear Dependence (3 points)

(version 1) Given that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \in \mathbb{R}^3$ and $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ 7 \\ \alpha \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

Find α such that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly dependent. Choose the best answer.

(a) 0
(b) 10
(c) Any scalar multiple of 10
(d) Any real number
(e) There is no value such that the vectors are linearly dependent

(version 2) Given that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \in \mathbb{R}^3$ and $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 10 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ 7 \\ \alpha \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

Find α such that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly dependent. Choose the best answer.

(a) 0
(b) 20
(c) Any scalar multiple of 20
(d) Any real number
(e) There is no value such that the vectors are linearly dependent

8. Eigenspace and Nullspace. (4 points)

Given $\alpha \in \mathbb{R}$, $\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 2-\alpha & 1 & 2 \\ 0 & 1-\alpha & 4 \\ 0 & 0 & 2-\alpha \end{bmatrix}$, if there exists a vector $\vec{x}$ such that $\mathbf{B}\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$, which of the following are true?

(a) $\text{rank}(\mathbf{A}) = 3$

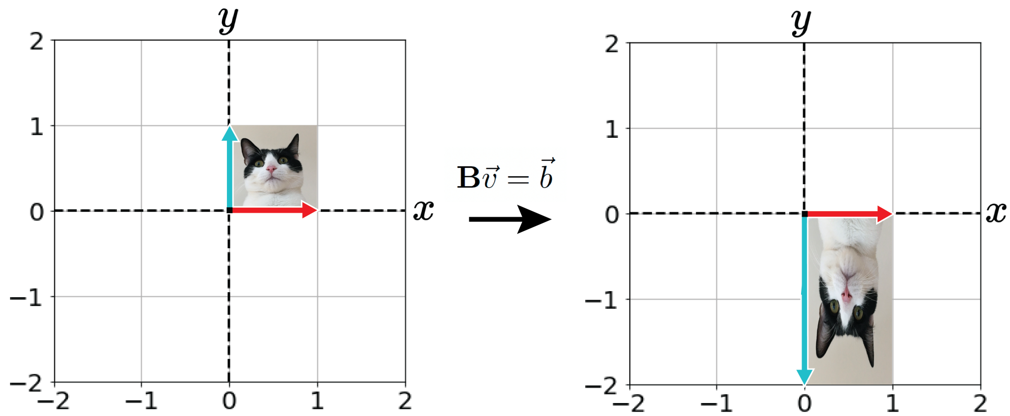
(b) $\vec{x}$ is in an eigenspace of **B**.

(c) $\vec{x}$ is in the null space of **B**.

(d) $\vec{x}$ is in an eigenspace of **A**.

9. Transition matrix Part I (2 points) You want to practice some of the matrix transformation skills you've learned in EECS 16A, and decide to try them out on some cat photos.

Choose the correct matrices to transform the photo on the left to the photo on the right. Assume $\vec{v}$ represents a vector pointing to any pixel in the image, and $\vec{b}$ is the resulting vector after applying the transformation matrix to $\vec{v}$.



$\mathbf{B}\vec{v} = \vec{b}$, where $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Which matrix $\mathbf{B}$ will result in the transformation seen above?

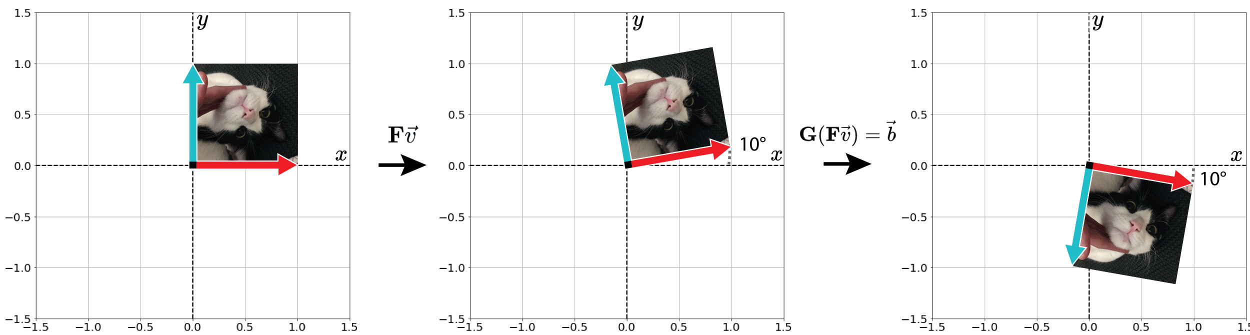
(a) $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

10. Transition matrix Part II (3 points)



$\mathbf{G}(\mathbf{F}\vec{v}) = \vec{b}$, where $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \vec{b} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$.

Which matrices **G** and **F** will result in the transformation seen above?

(a) Choose matrix **G** and **F** from the following matrices.

$\mathbf{M}1 = \begin{bmatrix} \cos(-10^\circ) & -\sin(-10^\circ) \\ \sin(-10^\circ) & \cos(-10^\circ) \end{bmatrix}$,

$\mathbf{M}2 = \begin{bmatrix} \cos(10^\circ) & \sin(10^\circ) \\ -\sin(10^\circ) & \cos(10^\circ) \end{bmatrix}$,

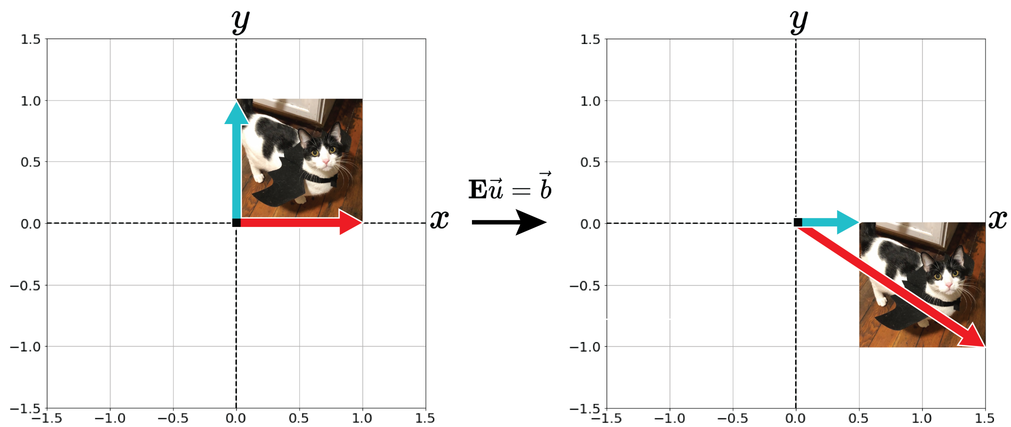
$\mathbf{M}3 = \begin{bmatrix} \cos(10^\circ) & -\sin(10^\circ) \\ \sin(10^\circ) & \cos(10^\circ) \end{bmatrix}$,

$\mathbf{M}4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{M}5 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{M}6 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) Would $\mathbf{F}(\mathbf{G}\vec{v})$ result in the same vector $\vec{b}$?

11. Transition matrix Part III (3 points)

Expressing translation in matrix multiplication



$\vec{b} = \vec{v} + \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}$, where $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$.

Consider $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$, we can express the translation in matrix multiplication form: $\mathbf{E}\vec{u} = \vec{b}$.

Which matrix $\mathbf{E}$ will result in the transformation seen above?

(a) $\begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & -1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0.5 \\ 0 & -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$

(e) such matrix $\mathbf{E}$ does not exist

12. Null space (4 points) Which of the following vectors are in the null space of matrix $\mathbf{A}A$? (Select all that apply)

(version 1) $\mathbf{A} = \begin{bmatrix} 1 & 5 & 8 & 9 \\ 2 & 4 & 10 & 12 \end{bmatrix}$

(a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

(version 2) $\mathbf{A} = \begin{bmatrix} 8 & 9 & 1 & 5 \\ 10 & 12 & 2 & 4 \end{bmatrix}$

(a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 8 \\ 10 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}$

13. Null space dimension (3 points)

How many basis vectors are needed to span the null space of matrix $\mathbf{B}$?

(version 1) $\mathbf{B} = \begin{bmatrix} 2 & 1 & 3 & 0 & 8 \\ 4 & 2 & 6 & 0 & 16 \\ 6 & 3 & 9 & 0 & 24 \\ 8 & 4 & 12 & 0 & 32 \end{bmatrix}$

(version 2) $\mathbf{B} = \begin{bmatrix} 2 & 1 & 4 & 0 & 8 \\ 4 & 2 & 8 & 0 & 16 \\ 6 & 3 & 12 & 0 & 24 \\ 8 & 4 & 16 & 0 & 32 \end{bmatrix}$

14. Null space, rank and linear independence (4 points)

Suppose $\vec{u}$ and $\vec{v}$ are unique vectors ($\vec{u} \neq \vec{v}$) in the null space of a square matrix $\mathbf{D}$, and vector $\vec{w} = \vec{u} - \vec{v}$.

For each of the following statements, choose either "True", "False", or "Not enough information".

(a) Matrix $\mathbf{D}$ is not full-rank.

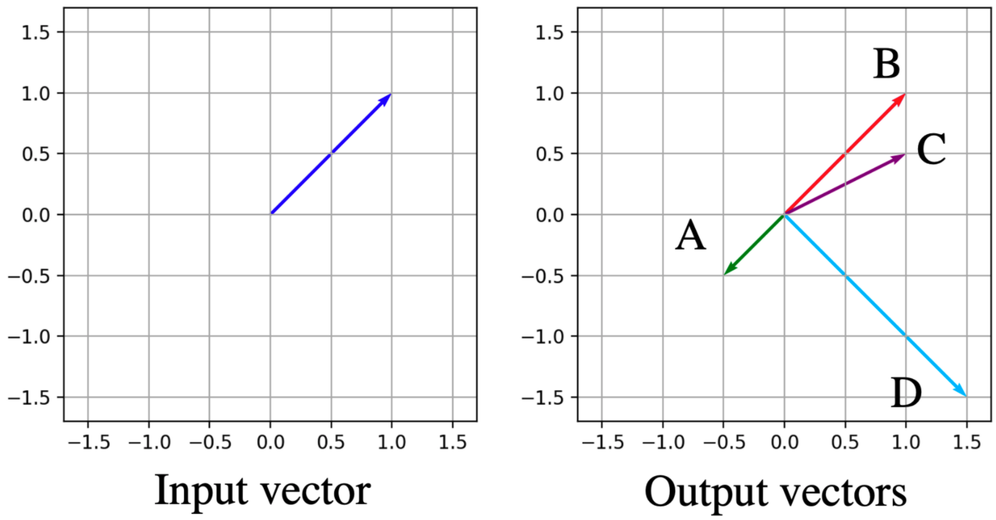
(b) Vector $\vec{w}$ is in the null space of matrix $\mathbf{D}$.

(c) Vectors =$\vec{u}$ and $\vec{v}$ are linearly independent.

15. Eigenvector. Bcourses Question 15

A 2×2 matrix $\mathbf{A}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0.5$.

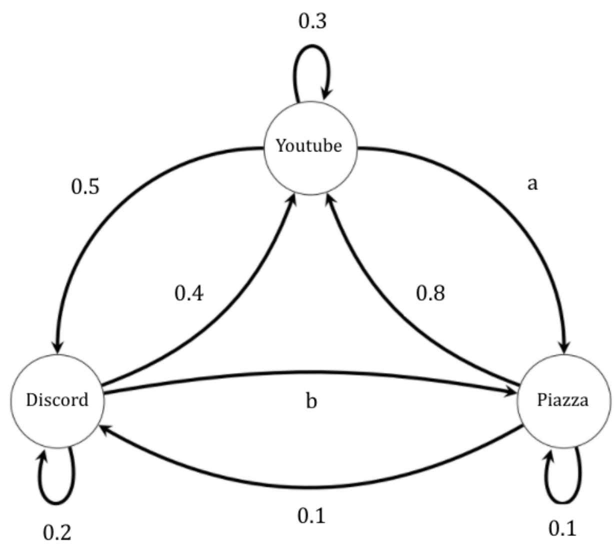
In the left figure, we have a vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Given $\vec{v}$ is an eigenvector of $\mathbf{A}$ and $\vec{u} = \mathbf{A}\vec{v}$, which vector(s) in the right figure is/are a possible $\vec{u}$ (Select all that apply)



16. Transition matrix Part I (7 points)

Prof. Arias decides to study the internet behavior of EECS 16A students on a typical weekday night. The number of students on the top three websites (Youtube, Piazza, and Discord) at time LaTeX: tt can be expressed as follows: $x_y[t], x_p[t], x_d$, respectively.

She finds that the flow of students across the three websites can be shown as follows:



(a) Let $\vec{x}[t] = \begin{pmatrix} x_y[t] \\ x_p[t] \\ x_d[t] \end{pmatrix}$ where $x_y[t], x_p[t], x_d[t]$ represent the number of students on Youtube, Piazza, and Discord at time t. Determine $\mathbf{A}$ such that $\vec{x}[t+1] = \mathbf{A}\vec{x}[t]$.

(b) Determine a and b such that the system is conservative.

17. Transition matrix Part II (12 points)

You are given the following transition matrix $\mathbf{B}$, from another study. There are in total 1200 students.

(version 1)$\mathbf{B} = \begin{pmatrix} 0.5 & 0 & 1 \\ 0 & 0.8 & 0 \\ 0.5 & 0.2 & 0 \end{pmatrix}$

(version 2)$\mathbf{B} = \begin{pmatrix} 0.5 & 0 & 1 \\ 0 & 0.4 & 0 \\ 0.5 & 0.6 & 0 \end{pmatrix}$

(c) Assume that $\vec{x}[t+1] = \mathbf{B}\vec{x}[t]$. If $\vec{x}[2]$ is measured to be $\begin{pmatrix} 600 \\ 160 \\ 440 \end{pmatrix}$, determine the state vector at the previous timestep $\vec{x}[1]$.

(d) You are given that one of the eigenvalues of $\mathbf{B}$ is $\lambda = 1$. Determine its corresponding eigenvector. Scale your solution such that the last element in the vector is 1.

(e) From part (c), we know that $\vec{x}[2] = \begin{pmatrix} 600 \\ 160 \\ 440 \end{pmatrix}$. After infinite time points, what is the number of students on each website? That is, find $\vec{x}[\infty]$.

18. A proof in 2 steps Part I (5 points) You are given two matrices, $A, B \in \mathbb{R}^{N \times N}$. In Question 18-19, we try to prove the following theorem in 2 steps. Please read through both questions before answering Question 18.

Theorem: If the columns of A are linearly dependent, then the columns of (AB) are also linearly dependent.

Case: 1

(1) Consider the case where [choose from below]

(a) A is not invertible
(b) A is invertible
(c) B is non-invertible
(d) B is invertible

(2) Based on [choose from below] there exists $\vec{u} \neq \vec{0}$ such that $B\vec{u} = 0$

(a) the theorem if-statement
(b) (1)

(3) $(AB)\vec{u} = A(B\vec{u}) = 0$ because of [choose from below]

(a) distributivity property
(b) associativity property
(c) 'A' having a trivial null-space
(d) 'A' having a non trivial null-space

Hence, the columns of (AB) are linearly dependent in this case.

19. A proof in 2 steps Part II (6 points)

Case: 2

(1) Consider the case where [choose from below]

(a) A is invertible
(b) A is non invertible
(c) B is invertible
(d) B is non invertible

(2) We [choose from below] there exists $\vec{v} \neq \vec{0}$ such that $A\vec{v} = \vec{0}$

(a) assume
(b) know

(3) Because of (1), [choose from statements a-d]

(4) Therefore, [choose from statements a-d]

Hence, the columns of (AB) are linearly dependent in this case.

Bank of statements you can use in (3) and (4):

(a) There exists $\vec{v} \neq \vec{0}$ such that $B\vec{u} = \vec{v}$

(b) There exists $\vec{u} \neq \vec{0}$ such that $B\vec{u} = \vec{v}$

(c) $(AB)\vec{u} = A(B\vec{u}) = A\vec{v} = 0$

(d) $(AB)\vec{v} = (BA)\vec{v} = B(A\vec{v}) = 0$