1. DFT

Determine the DFT for each signal described below.

(a) $x(n) = \sin\left(\frac{2\pi}{5}n\right) + \cos\left(\frac{4\pi}{5}n\right), \quad \forall n$, with window of length 5

Rewrite: $x(n) = \frac{e^{j2\pi/5 n} - e^{-j2\pi/5 n}}{2j} + \frac{e^{j4\pi/5 n} + e^{-j4\pi/5 n}}{2}$

$= -\frac{1}{2}j e^{j2\pi/5 n} + \frac{1}{2}j e^{j8\pi/5 n} + \frac{1}{2} e^{j4\pi/5 n} + \frac{1}{2} e^{j6\pi/5 n}$

$k=1 \quad k=4 \quad k=2 \quad k=3$

So $X_k = \begin{bmatrix} 0 \\ -\frac{5}{2}j \\ \frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2}j \end{bmatrix}$ Because $x(n) = \sum_{k=0}^{N-1} X_k \frac{e^{ik\omega_0 n}}{N}$!

(b) $x(n) = \delta(n) - \delta(n-2)$, with window of length 6

$\delta(n) \longleftrightarrow X_k = 1$ $\omega_0 = \pi/3$

Time-shifting property:

$\delta(n-2) \longleftrightarrow X_k e^{-ik\omega_0(2)} = e^{-ik2\pi/3}$

So: $X_k = X_{k_{\delta(n)}} - X_{k_{\delta(n-2)}} = 1 - e^{-ik2\pi/3}$

2. Matrix Multiplication Suppose $x(n)$ is the input signal applied to a linear time-invariant (LTI) system characterized by the impulse response $h : \mathbb{Z} \to \mathbb{R}$ and corresponding frequency response H, where

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n}, \quad \forall \omega$$

Let $y(n)$ be the corresponding output signal. If

$$h(n) = \frac{\delta(n) + \delta(n-10)}{2}$$

and

$$x(n) = \sin\left(\frac{2\pi}{10}n\right) + \cos\left(\frac{2\pi}{45}n\right),$$

determine an appropriate DFT window size that is the same length as the period of $x(n)$. Then find the output DFT coefficients Y_k in terms of the input DFT coefficients X_k.

An appropriate window size would be 90 (the LCM of the periods).
(Note that coefficients will only be present at $k = \pm 10, \pm 45$.)

Based on the same time-shifting property used in (b),

$$H_k = 1/2 + 1/2\, e^{-ik\omega_0 (10)} = 1/2 + 1/2\, e^{-ik2\pi/9}$$

Then $Y_k = \left[1/2 + 1/2\, e^{-ik2\pi/9}\right] X_k$

3. Symmetric Matrices

Prove the following: For any symmetric matrix A, any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Hint: Use the definition of an eigenvalue to show that $\lambda_1(v_1 \cdot v_2) = \lambda_2(v_1 \cdot v_2)$.

$$\begin{aligned}\lambda_1(v_1 \cdot v_2) &= Av_1 \cdot v_2 \\ &= (Av_1)^T v_2 \\ &= v_1^T A^T v_2 \\ &= v_1^T A v_2 \\ &= v_1^T(\lambda_2 v_2) \\ &= \lambda_2(v_1 \cdot v_2)\end{aligned}$$

So $\lambda_1(v_1 \cdot v_2) - \lambda_2(v_1 \cdot v_2) = 0$

$(\lambda_1 - \lambda_2)(v_1 \cdot v_2) = 0$

But $\lambda_1 \neq \lambda_2$, so $v_1 \cdot v_2 = 0$. ∎