1. Proof of Induction

Given the voltage-current relationship of an inductor \( V = L \frac{di}{dt} \), show that its complex impedance is \( Z_L = j\omega L \).

**Solutions:** Since \( V(t) \) is an eigenfunction and impedance is a constant, then \( i(t) \) is also an eigenfunction such that \( \frac{di}{dt} = j\omega \cdot i(t) \).

\[
\begin{align*}
V &= L \frac{di}{dt} \\
V &= L j\omega \cdot i(t) \\
V &= (j\omega L) i(t) \\
Z_L &= j\omega L
\end{align*}
\]

2. L-R Filter

![L-R Filter Diagram]

(a) Write the frequency response function \( H(\omega) = \frac{V_{out}}{V_{in}} \) for the circuit.

**Solutions:** \( H(\omega) = \frac{V_{out}}{V_{in}} = \frac{Z_R}{Z_L + Z_R} = \frac{R}{j\omega L + R} = \frac{1}{j\omega \frac{L}{R} + 1} \)

(b) If \( R = 10\Omega \) and \( L = 100\text{mH} \), plot the log-magnitude of \( H(\omega) \) and label important magnitudes and frequencies.

**Solutions:** \( \omega_L = \frac{R}{L} = 10/0.100 = 100\text{rad/s} \)
(c) If $V_{in}(t) = \cos(10t) + \cos(100t) + \cos(1000t)$, what is $V_{out}(t)$? (Assume that $H(-\omega) = H(\omega)$.)

**Caveat:** We have not yet discussed how the phase of $V_{in}$ is affected by the circuit. Ignore this for now; assume the phase of the output is the same as the phase of the input.

**Solutions:**

$$H(\omega) = \frac{1}{\sqrt{\omega^2 + \frac{L^2}{R^2}}} = \frac{1}{\sqrt{\omega^2 + \frac{1}{10000}}}$$

$|H(10)| \approx 1$

$|H(100)| = \frac{1}{\sqrt{2}}$

$|H(1000)| = \frac{1}{10}$

$$V_{out} = 1 \cdot \cos 10t + \frac{1}{\sqrt{2}} \cdot \cos(100t) + \frac{1}{10} \cdot \cos(1000t)$$

3. **Op-Amps: What Can’t They Do?**

(a) Write the frequency response function $H(\omega) = \frac{V_{out}}{V_{in}}$ for the circuit.

**Solutions:** By Golden Rules:

$$V_o = V_p$$

$$i_n = 0$$

$$i_p = 0$$
\( V_n = 0 \) by the first golden rule, and using the second rule with KCL, we find that:

\[
\begin{align*}
\frac{V_{in}}{Z_R} &= -\frac{V_{out}}{Z_C} \\
\frac{V_{out}}{V_{in}} &= \frac{Z_C}{Z_R} \\
\frac{V_{out}}{V_{in}} &= -\frac{1}{j\omega RC} \\
H(\omega) &= -\frac{1}{j\omega RC}
\end{align*}
\]

(b) If \( R = 1k\Omega \) and \( C = 100nF \), plot the log-magnitude of \( H(\omega) \) and label important magnitudes and frequencies. (Again assume that \( H(-\omega) = H(\omega) \).)

\textbf{Solutions:}

(c) \textbf{Challenge:} What does this circuit do?

\textit{Hint:} You will probably need to set up the differential equation relating \( V_{out} \) and \( V_{in} \) to work this out.

\textbf{Solutions:}

\[
\frac{V_{in}}{R} = -C \frac{dV_{out}}{dt}
\]

\[-\frac{1}{RC} \int V_{in}dt = V_{out}\]

The circuit takes in \( V_{in} \) and integrates it.