DFT Recap

Synthesis Equation

\[ x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{ik\alpha_0 n} = \sum_{k=0}^{N-1} X_k \Psi_k(n) \]

Where \( \Psi_k(n) = \frac{1}{N} e^{ik\alpha_0 n} \) are basis function that capture the k’th harmonic

Analysis Equation:

\[ X_k = \sum_{k=0}^{N-1} x(n) e^{-ik\alpha_0 n} \]

\( X_k \) is N-periodic: \( X_k = X_{k+N} \)

\( \alpha_0 = \frac{2\pi}{N} \Rightarrow \alpha_0 N = 2\pi . \) Thus,

\[ X_{k+N} = \sum_{k=0}^{N-1} x(n) e^{-i(k+N)\alpha_0 n} = \sum_{k=0}^{N-1} x(n) e^{-ik\alpha_0 n} e^{-i2\pi n} = X_k \]

Two types of signals:

1. Signals periodic with period N can be fully represented by synthesis equation because the synthesis equation is also N periodic.

\[ x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{ik\alpha_0 (n+N)} = x(n) \]

2. From a long signal, take a window of N samples and represent that window with a DFT. Since this signal may or may not be periodic, the synthesis equation is only valid for values from 0 to N-1. In the real world, signals may not always be periodic, but we still care about the frequency components, so we take windows of length N.
Vector/Matrix Formulation

\[
< \Psi_k, \Psi_l > = \begin{bmatrix} \Psi_k(0) & \Psi_k(1) & \ldots & \Psi_k(N-1) \end{bmatrix} \begin{bmatrix} \Psi^*_l(0) \\ \Psi^*_l(1) \\ \vdots \\ \Psi^*_l(N-1) \end{bmatrix} = \begin{cases} \frac{1}{N}, & \text{if } k = l \\ 0, & \text{otherwise} \end{cases} = \frac{1}{N} \delta(k - l)
\]

\( \delta(n) \) is the Kronecker delta, the discrete-time impulse.

\[
X_0 = \sum x(n) e^{-i\omega_0 n} = \sum x(n)
\]

\[
X_k = \sum x(n) e^{-ik\omega_0 n} = \begin{bmatrix} x(0) & x(1) & \ldots & x(N-1) \end{bmatrix} \begin{bmatrix} 1 \\ e^{-i\omega_0} \\ \vdots \\ e^{-i(N-1)\omega_0} \end{bmatrix}
\]

If the frequency components form a vector, we can rewrite as the following matrix/vector product:

\[
\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & e^{-i\omega_0} & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-i\omega_0} & \ldots & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}
\]

Therefore

\[
X = Fx
\]

This analysis matrix represents a coordinate transformation.

How do I do synthesis?

\[
F^{-1} X = F^{-1} Fx
\]

Assuming \( F \) is invertible, then

\[
x = F^{-1} X
\]
\[ F^H F = F^T F = F^* F \]

Each row element is \( N \Psi_k \) so we have

\[
F^* F = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & e^{i\omega_0} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & e^{ik\omega_0} & \cdots & 1 \\
1 & \cdots & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & e^{-i\omega_0} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & e^{-ik\omega_0} & \cdots & 1 \\
1 & \cdots & \cdots & 1
\end{bmatrix}
= \begin{bmatrix}
\cdots \\
0 \\
0 & N^2 < \Psi_k, \Psi_k > \\
0 & \cdots \\
0
\end{bmatrix}
= \begin{bmatrix}
N \\
\cdots \\
0 \\
0 & N
\end{bmatrix} = NI
\]

Thus the inverse of \( F \) is

\[
F^{-1} = \frac{1}{N} F^H
\]

\[
x = \frac{1}{N} F^H X = \frac{1}{N} F^* X
\]

What happens when I do: \( x^H x \)

\[
x^H = \frac{1}{N} (F^H X)^H = \frac{1}{N} X^H F
\]

\[
x^H x = \frac{1}{N^2} X^H F F^H X = \frac{1}{N^2} N X^H X
\]

\[
x^H x = \frac{1}{N} X^H X
\]

Which can be rewritten as:

\[
\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |X_k|^2
\]

In other words, the energy of the signal is the same in the time domain and the frequency domain.
DFT Examples

Take a length-N DFT on $\delta(n)$, determine the $X_k$’s

$$X_k = \sum_{k=0}^{N-1} x(n)e^{-i\omega_0 kn}$$

$$X_k = x(0)e^{-i\omega_0 n}$$

$$X_k = 1$$

For all $k$, which means

$$\delta(n) = \frac{1}{N} \sum_{k=0}^{N-1} e^{ik\omega_0 n}$$

Every harmonic is present in an equal amount. Signals that are narrow in time tend to be wide in frequency

Example: DFT on $\delta(n-1)$

$$X_k = \sum_{k=0}^{N-1} x(n)e^{-i\omega_0 kn}$$

$$X_k = x(1)e^{-i\omega_0}$$

$$X_k = e^{-i\omega_0}$$

Shifting by 1 results in multiplying by $e^{-i\omega_0}$, thus

$$x(n-1) \leftrightarrow e^{-i\omega_0} \cdot 1$$

Example: Summing $\delta(n)$ with a length-4 DFT

$$\delta(n) = \frac{1}{4} \sum_{k=0}^{3} e^{ik\pi/2}$$

$$\delta(n) = \frac{1}{4} (1 + e^{i\pi/2} + e^{i\pi} + e^{i3\pi/2})$$

$$\delta(n) = \frac{1}{4} (1 + (-1)^n + i^n + (-i)^n)$$
Example: What is the period of \( x(n) = e^{in\pi/2} \)

\[ x(n + p) = x(n) \]
\[ x(n + p) = e^{i(n+p)\pi/2} \]
\[ x(n + p) = e^{in\pi/2} e^{ip\pi/2} \]

\[ \frac{p}{2} = 2\pi l \Rightarrow p = 4l \]

for \( l = 1, p = 4 \)

Example: Length-8 DFT on \( x(n) = e^{in\pi/2} \)

\[ \omega_0 = \frac{\pi}{4} \]

\[ X_k = \sum_{k=0}^{7} x(n)e^{-j k \omega_0 n} \]

\[ X_k = \sum_{k=0}^{7} e^{i \frac{\pi}{2} n} e^{-j k \frac{\pi}{4} n} \]

\[ X_k = \sum_{k=0}^{7} (e^{i \frac{\pi}{4}})^n \]

\[ X_k = \frac{e^{i(\frac{\pi}{4})8} - 1}{e^{i(\frac{\pi}{4})} - 1} \]

Therefore when \( k = 2, X_k = 8 \) and when \( k \neq 2, X_k = 0 \).

By inspection we could also notice that

\[ e^{i \frac{\pi}{2} n} = e^{i2 \frac{\pi}{4} n} \]

Which corresponds to a single frequency at \( f = 2\omega_0 \)