Overview

1. Phase response and its importance
2. Geometric interpretation of the phase/magnitude reponse
3. Low pass and high pass filter examples

Magnitude Response

Recall our simple low pass filter response:

\[ H(\omega) = \frac{1}{1 + j\omega RC} \]

We can find the magnitude by:

\[ ||H(\omega)|| = \frac{1}{\sqrt{1 + (\omega RC)^2}} \]

What happens when \( \omega = \frac{1}{RC} \)? The response becomes \( H(\omega) = \frac{1}{\sqrt{2}} = -3dB \). Also know was the cutoff frequency since most of everything \( \omega < \frac{1}{RC} \) passes through but \( \omega > \frac{1}{RC} \) is attenuated.
Now suppose $\omega >> \frac{1}{RC}$,

$$||H(\omega)|| \approx \frac{1}{\omega RC}$$

This is a first order filter since as $\omega$ increases, the magnitude decreases as $\frac{1}{\omega}$. Similarly for second order, the magnitude decreases as $\frac{1}{\omega^2}$.

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An ideal low pass filter would simply be a "box" filter. All frequencies below our cutoff frequency would remain untouched, while all frequencies above the cutoff would be infinitely attenuated. An ideal low pass (antialiasing) filter would be infinite order, and thus would not be physically implementable.

How can we figure out how well we can filter out unwanted signals? Suppose we have a signal of certain $\omega_0$. We can look on the graph at $\omega_0$ to determine the attenuation and error from that signal. For a second order filter, we will have more attenuation and less error.
Phase Response

Due to the imaginary part, filters can also change phase. Recall that:

\[ e^{j\theta} = \cos(\theta) + j\sin(\theta) \]

Thus multiplying by some complex number corresponds to a phase shift:

\[ e^{j\theta} \cdot e^{j\omega} = e^{j(\omega + \theta)} \]

Why is phase response important? Suppose we have two signals of \( \omega_1 \) and \( \omega_2 \) such that

\[ \omega_1 = 2\omega_2 \]

A \( 90^\circ \) phase shift in \( \omega_1 = \frac{2\pi}{T} \) corresponds to a \( \frac{T}{4} \) shift in time. However for \( \omega_2 \) we have a \( \frac{T}{8} \) time shift. Therefore, even if a signal is not attenuated, the shift can cause changes in the output signal. This can become a problem in feedback systems and controls (which we will see later in the course).

Consider the RC circuit:

We have the equation:

\[ RC\dot{y}_c(t) + y_c(t) = x(t) \]

To determine \( H_c(\omega) \), let the input be \( x(t) = e^{j\omega t} \) and then

\[ y_c(t) = H_c(\omega)e^{j\omega t} \]
\[ \dot{y}_c(t) = i\omega H_c(\omega)e^{j\omega t} \]

Here \( \dot{y}_c(t) \) denotes the time derivative of \( y_c(t) \)

Plug into our differential equation:

\[ RCi\omega H_c(\omega)e^{j\omega t} + H_c(\omega)e^{j\omega t} = e^{j\omega t} \]

\[ H_c(\omega) = \frac{1}{i\omega RC + 1} \]
Plotting the Magnitude Response

Suppose we have

\[ H_c(\omega) = \frac{1}{i\omega RC + 1} \]

If we factor out \( RC \), then

\[ H_c(\omega) = \frac{1}{RC} \frac{i\omega + \frac{1}{RC}}{i\omega - \left( -\frac{1}{RC} \right)} \]

Recall that the difference of two complex number \( z_1 + z_2 \) on the complex plane is:

\[ z_1 - z_2 \]

Now lets view our frequency response on the complex plane:

\[ |H_c(\omega)| = \frac{1}{|RC|} \]

where \( blue \) is the blue vector shown above.
Based off the geometry, we can tell that the magnitude response at $\omega = 0$ is a maximum of 1.

Why does the magnitude response decrease as $\omega \geq 0$ increases? The length of the blue vector increases, so the ratio decreases.

Now for curvature, slope is slow changing near 0 and $\infty$, since the length of the vector doesn’t change as much. In the intermediate range, we have an inflection point.

When do we get a magnitude response of $\frac{1}{\sqrt{2}}$? When $\omega = \frac{1}{RC}$, we have a right triangle corresponding to a factor of $\frac{1}{\sqrt{2}}$. 

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Plotting the Magnitude Response

Recall that
\[
\angle \frac{z_1}{z_2} = \angle z_1 - \angle z_2
\]

Thus:
\[
\angle H_c(\omega) = \angle \left( \frac{1}{RC} \right) - \angle |\text{blue}| \\
\angle H_c(\omega) = 0 - \angle |\text{blue}|
\]

Which corresponds to the negative arctan:
\[
\angle H_c(\omega) = -\arctan \left( \frac{\omega}{1/RC} \right)
\]
RC Low Pass Filter

Recall the RC circuit:

\[ x(t) = y_R(t) + y_C(t) \]
\[ e^{i\omega t} = H_R(\omega)e^{i\omega t} + H_C(\omega)e^{i\omega t} \]
\[ 1 = H_R(\omega) + H_C(\omega) \]
\[ 1 = H_R(\omega) + \frac{1}{i\omega RC + 1} \]
\[ H_R(\omega) = 1 - \frac{1}{i\omega RC + 1} \]
\[ H_R(\omega) = \frac{i\omega RC + 1 - 1}{i\omega RC + 1} \]
\[ H_R(\omega) = H_R(\omega) = \frac{i\omega RC}{i\omega RC + 1} \]
\[ H_R(\omega) = \frac{i\omega}{i\omega + \frac{1}{RC}} \]