Recap: State-space model:

\[
\begin{align*}
\text{open-loop:} & \\
\text{closed-loop:}
\end{align*}
\]

\[
\begin{align*}
U(k) & \rightarrow x(k+1) = Ax(k) + Bu(k) \rightarrow y(k) \\
& \text{or} \\
& r(k) \rightarrow K \rightarrow U(k) \rightarrow y(k)
\end{align*}
\]

(repeat derivation from previous):

\[
\begin{align*}
x(k+1) &= (A - BK) x(k) + BK r(k) \\
y(k) &= C x(k)
\end{align*}
\]

- Open-loop and closed-loop models take the same structure, so the analysis generally applies to both:

\[
X(k) = A^k x(0) + A^{k-1} Bu(0) + A^{k-2} Bu(1) + \ldots + ABu(k-2) + Bu(k-1)
\]

- Stability depends on the e-vals of $A$.
  - $\lambda, \nu$ could be real or complex (but if $\lambda = \alpha + j \beta$, $\nu = e^{\alpha t} j e^{j \beta t}$, then if $A$ is real, $\lambda = \alpha - j \beta$, $\nu = e^{\alpha t} e^{-j \beta t}$ are also e-vals e-vecs of $A$).

- Define stability: $X(k)$ converges to a "finite" vector $X_{\text{ref}} = \begin{bmatrix} x_{\text{ref}} \\ \eta_{\text{ref}} \end{bmatrix}$ as $k \to \infty$. ($X(k)$ does not go unbounded.)

**Internal stability:** All eigenvalues of $A$ have magnitude $< 1$. That is, they fall within the unit circle. This is the stability of the system without input.

**Bounded input, bounded output (BIBO) stability.** Only those eigenvalues of $A$ that appear in the input-output transfer function must have magnitude $< 1$. 

Ex.: 
\[ x(k+1) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x(k) + Bu(k) \quad y(k) = C x(k) \]

- Internally stable if \( |\lambda_1| < 1, |\lambda_2| < 1 \)
- BIBO stability:
  - If \( B = [I] \), then the input only affects \( y \), so we could have \( |\lambda_1| \)
  - and still have BIBO stability if \( C = [0 \ 0] \). Even if \( y \) were
    - unstable it would be hidden from the output.

**Case A:** Input is 0, \( x(k) = A^k x(0) \)

1. \( A \) is diagonalizable \( \Rightarrow A \) has n linearly independent eigenvectors, 
   \( \Lambda = T A T^{-1} \) columns are eigenvectors of \( A \) 
   diagonal matrix with the eigenvalues of \( A \)

Ex: \( x(0) = \sigma_1 v_1 + \sigma_2 v_2 \), where \( v_1, v_2 \) are e-vectors of \( A \) 

Then \( x(k) = \sigma_1 A^k v_1 + \sigma_2 A^k v_2 = \sigma_1 \lambda_1^k v_1 + \sigma_2 \lambda_2^k v_2 \)

**Note that if** \( |\lambda_i| < 1 \), the \( x(k) \rightarrow 0 \! ! \)
- If \( |\lambda_i| > 1 \) (either one, both), \( x(k) \rightarrow \infty \)
- If \( |\lambda_i| = 1 \) (both), \( \| x(k) \| = \| x(0) \| \) ("marginally stable")

- Response goes to 0 immediately.

**Case (a):** \( \lambda_1 = \lambda_2 = 0 \)
- Response goes to 0 immediately.

**Case (b):** \( \lambda_1 = \lambda_2 = 0.5 \)
- Response is halved each time

**Case (c):** \( \lambda_1 = \lambda_2 = -0.5 \)
Case (d): \[ \lambda_i = \lambda_2 = -1 \]

Case (e): \[ \lambda_i = d_2 = 1 \]