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Lecture given by Prof. Claire Tomlin

1. Recap

open-loop:
\[ x(k+1) = Ax(k) + Bu(k) \]
\[ y(k) = Cx(k) \]

closed-loop:
\[ x(k+1) = A_{CL}x(k) + B_{CL}y_d(k) \]
\[ x(k+1) = (A - BKC)x(k) + (BK)y_d(k) \]
\[ y(k) = Cx(k) \]

For a given initial condition vector \( x_0 \) and \( k \) inputs:
\[ x(k) = A^k x_0 + \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-2) \\ u(k-1) \end{bmatrix} \begin{bmatrix} \sigma_1 v_1 \\ \sigma_2 v_2 \\ \sigma_1 \lambda_1^{k-1} v_1 + \sigma_2 \lambda_2^{k-1} v_2 \end{bmatrix} \]

2. System Behavior - Zero Input

Consider a zero input system. \( u(k) = 0 \ \forall \ k \).
The matrix \( A \) in the system is diagonalizable, meaning that \( A \) has \( n \) linearly independent eigenvectors.

Let \( n = 2 \):
\[ x_0 = \sigma_1 v_1 + \sigma_2 v_2 \]
\[ x(k) = A^k (\sigma_1 v_1 + \sigma_2 v_2) \]
\[ x(k) = \sigma_1 \lambda_1^k v_1 + \sigma_2 \lambda_2^k v_2 \]

We can determine the system’s behaviour for a given input from a geometric point of view. We look at \( x_1 \) and \( x_2 \), the state variables of a system and plot the trajectories of \( (x_1(k), x_2(k)) \) for any given \( k \) along with their eigenvectors \( v_1 \) and \( v_2 \). If \( A \) is diagonal, then the eigenvectors \( v_1 \) and \( v_2 \) become the \( x_1 \) and \( x_2 \) axes. In general when \( A \) is diagonalizable, they are not these axes but we know they are linearly independent. When the initial condition is a scaled version of one of the eigenvectors, then the trajectory will always follow the eigenvector (tending towards 0 if the eigenvalue is less than 1 and towards \( \infty \) otherwise). Consider the following cases for \( \lambda_1 \) and \( \lambda_2 \):

Case 1:
\[ \lambda_1, \lambda_2 \in \mathbb{R} \]
\[ \lambda_1 = \lambda_2 > 0 \]
\[ |\lambda_1| < 1, |\lambda_2| < 1 \]

Since the eigenvalues are equal, \( x(k) \) heads toward the origin in a linear fashion.

**Case 2:**
\[ \lambda_1, \lambda_2 \in \mathbb{R} \]
\[ \lambda_2 > \lambda_1 > 0 \]
\[ |\lambda_1| < 1, |\lambda_2| < 1 \]

Since \( \lambda_2 > \lambda_1 \), \( x(k) \) heads toward the origin but also tends toward \( v_2 \) asymptotically like a hyperbolic curve.

**Case 3:**
\[ \lambda_1, \lambda_2 \in \mathbb{R} \]
\[ \lambda_1 > 0 > \lambda_2 \]
\[ |\lambda_1| < 1, |\lambda_2| < 1 \]

Since \( \lambda_2 < 0 \), \( x(k) \) bounces from one side to the other.

**Case 4:**
\[ \lambda_1, \lambda_2 \in \mathbb{R} \]
\[ \lambda_1 > 0, \lambda_2 > 0 \]
\[ |\lambda_1| > 1 > |\lambda_2| \]

Since \( |\lambda_1| > 1 \), \( x(k) \) will veer off to infinity and the origin acts like a saddle.

**Case 5:**

- \( \lambda_1, \lambda_2 \in \mathbb{C} \)
- \( \lambda_1 = \lambda_2^* \)
- \( |\lambda_1| < 1, |\lambda_2| < 1 \)

Because the eigenvalues and eigenvectors are complex conjugate pairs, we look at the real and imaginary components.
of the eigenvectors instead. Consider:

\[ v_1 = e_1 + je_2 \]
\[ v_2 = e_1 - je_2 \]
\[ |\rho| < 1 \]
\[ \lambda_1 = \rho e^{j\theta} \]
\[ \lambda_2 = \rho e^{-j\theta} \]
\[ \alpha = \alpha_1 + j\alpha_2 \]
\[ x_0 = \alpha v_1 + \alpha^* v_2 \]
\[ = \alpha e_1 + \alpha_2 e_2 \]
\[ = \alpha_1 \text{Re}(v_1) + \alpha_2 \text{Im}v_1 \]
\[ x(k) = A^k x_0 \]
\[ x(k) = \alpha_1 \text{Re}(A^k v_1) + \alpha_2 \text{Im}A^k v_1 \]
\[ x(k) = \alpha_1 \text{Re}(\lambda^k v_1) + \alpha_2 \text{Im}\lambda^k v_1 \]
\[ x(k) = \alpha_1 \text{Re}(\rho^k (\cos(\theta k) + j\sin(\theta k))(e_1 + je_2)) + \alpha_2 \text{Im}(\rho^k (\cos(\theta k) + j\sin(\theta k))(e_1 + je_2)) \]
\[ x(k) = \alpha_1 \rho^k \cos(\theta k)e_1 + \alpha_2 \rho^k \sin(\theta k)e_2 = \alpha_1 \rho^k \sin(\theta k)e_1 - \alpha_1 \rho^k \sin(\theta k)e_2 + \alpha_2 \rho^k \cos(\theta k)e_2 \]