Stability in Closed-Loop Systems

To recap, we define our closed-loop system with the following constraints:

\[ x[k+1] = A_{CL}x[k] + B_{CL}r[k] \]
\[ y[k] = Cx[k] \]

In the above system, we define \( A_{CL} = A - BKC \) and \( B_{CL} = BK \), where \( A, B, C, \) and \( K \) are defined as the corresponding matrix parameters of the corresponding open-loop controller to our given system.

Our goal is to make our closed-loop system stable (and hence dynamically able to withstand environmental fluctuations), and this is a goal achievable by intelligent selection of our gain matrix \( K = [k_1 \ k_2] \). We know that in order for the system to be stable, all \( |\lambda_i| < 1 \), a constraint which carries with it an implication of convergence towards a steady-state.

Now let’s dive into the mathematics used to optimize our gain matrix selection - assume for simplicity that \( A = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix} \), \( B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), and \( C = \begin{bmatrix} 1 & 0 \end{bmatrix} \). This is done without loss of generality - the results of the following derivation can be easily extrapolated to other configurations of \( A, B, \) and \( C \). Further assume for now that our gain matrix is simply \( 1 \times 1 \) - or rather, a scalar \( K \). Substituting \( A_{CL} = A - BKC \), we yield
\[ A_{CL} = \begin{bmatrix} \lambda_1 & 1 \\ -K & \lambda_2 \end{bmatrix} \]. Computing and solving the characteristic polynomial of this \(2 \times 2\) matrix yields the following expression for the eigenvalues of \(A_{CL}\):

\[ \lambda_{CL} = \frac{(\lambda_1 + \lambda_2) \pm \sqrt{(\lambda_1 + \lambda_2)^2 - 4(\lambda_1 \lambda_2 + K)}}{2} \]

We now define the concept of the "root locus" - all the values of \(\lambda_{CL}\) as we vary \(K\). Let’s make an additional assumption of simplicity that \(\lambda_1 = \lambda_2 = 1\), yielding \(\lambda_{CL} = \frac{1 \pm \sqrt{4 - 4(1 + K)}}{2}\), which implies that in many instances, the resulting eigenvalues of the system will be complex - a phenomenon we wish to avoid. This is easy - we simply select values of \(K\) that are negative, and we get positive eigenvalues.

Now, if we repeat this process with a two-input system - where our gain matrix \(K\) is now defined as a \(1 \times 2\) matrix, \([k_1 \ k_2]\), our system becomes significantly more tricky to optimize. We can do this by first repeating the eigenvalue computation for \(A_{CL}\) as we did above. Solving for \(A_{CL}\) when \(K\) is \(2 \times 1\) yields

\[ A_{CL} = \begin{bmatrix} \lambda_1 & 1 \\ -k_1 & \lambda_2 - k_2 \end{bmatrix} \],

which has a corresponding characteristic equation as below:

\[ \lambda_{CL}^2 + (k_2 - \lambda_1 - \lambda_2)\lambda_{CL} + (k_1 - \lambda_1 k_2 + \lambda_1 \lambda_2) = 0 \]

The optimal eigenvalues to have for our system are the case when both values of \(\lambda_{CL} = 0.5\), which is obtained when the corresponding characteristic equation is \((\lambda_{CL} - 0.5)^2 = 0\), or \(\lambda_{CL} - \lambda_{CL} + 0.25 = 0\). With our grouping of terms as above, we now have a system of two equations to solve:

\[ k_2 - \lambda_1 - \lambda_2 = -1 \]

\[ k_1 - \lambda_1 k_2 + \lambda_1 \lambda_2 = 0.25 \]

Simply solving the above system for \(k_1\) and \(k_2\) will yield the optimal eigenvalues for our closed-loop system.