Notes

Symmetric Matrices

A matrix T is called symmetric if $T = T^*$.

Positive (Semi-) Definite Matrices

A matrix T is a positive semi-definite matrix if it is symmetric and,

$$v^*Tv \geq 0 \text{ for all } v \in \mathbb{C}^n$$

Additionally, it is positive definite if,

$$v^*Tv = 0 \text{ if and only if } v = 0$$

Complex Spectral Theorem: Statement

Let T be a symmetric matrix from $\mathbb{C}^n$ to $\mathbb{C}^n$. Then,

(a) There exists n linearly independent eigenvectors of T that form a basis for $\mathbb{C}^n$. Further more, the eigenvectors are orthonormal.

(b) The eigenvalues of T are real.

Questions

1. **Eigenvalues are Real**

   Prove the following: For any symmetric matrix A, any eigenvalue of A is real.

   Hint: Use the definition of an eigenvalue to show that $\lambda^*(\vec{v}^*\vec{v}) = \lambda(\vec{v}^*\vec{v})$.

2. **Eigenvectors are Orthogonal**

   Prove the following: For any symmetric matrix A, any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

   Hint: Use the definition of an eigenvalue to show that $\lambda_1(\vec{v}_1^T\vec{v}_2) = \lambda_2(\vec{v}_1^T\vec{v}_2)$.

3. **Power Iteration**

   Power iteration is a method for approximating eigenvectors of a matrix A numerically. It's particularly effective when A is very large but very sparse. For example Google's PageRank algorithm, used to determine the ranking of search results, essentially attempts to perform power iteration on the adjacency matrix of links between all web pages on the internet.

The method starts with any vector x^0 and then iterates the following update:

$$\vec{x}^{k+1} = \frac{A\vec{x}^k}{||A\vec{x}^k||}.$$

You will show that this algorithm converges for symmetric A.

(a) Show that if A is a diagonal matrix $\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ with λ_1 strictly greater than the other λ_i then the power iteration method converges to $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, which is the eigenvector corresponding to the largest eigenvalue of A.

(b) Now use the spectral decomposition to show that for any symmetric matrix whose largest eigenvalue is strictly greater than its other eigenvalues, the power iteration method converges to the eigenvector corresponding to the largest eigenvalue.

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