EECS 16B Designing Information Devices and Systems II
Fall 2016 Murat Arcak and Michel Maharbiz Discussion 1B

# 1 Complex Numbers

A complex number $z$ is an ordered pair $(x, y)$, where $x$ and $y$ are real numbers, written as $z = x + iy$ such that $i = \sqrt{-1}$. The magnitude of a complex number $z = a + ib$ is denoted as $|z|$ and is given by,

$$|z| = \sqrt{x^2 + y^2}$$

The phase or argument of a complex number is denoted as $\theta$ and is defined to be,

$$\theta = \text{atan2}(x, y)$$

Here, $\text{atan2}(x, y)$ is a function that returns the angle from the positive x-axis to the vector from the origin to the point $(x, y)$. A complex number can also be written in polar form as follows.

$$z = |z|e^{i\theta}$$

Euler's Identity is,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

With this definition, the polar representation of a complex number will make more sense. Note that,

$$|z|e^{i\theta} = |z|\cos(\theta) + i\,|z|\sin(\theta)$$

The reason for these definitions is to exploit the geometric interpretation of complex numbers, as illustrated in Figure 1, in which case $|z|$ is the magnitude and $e^{i\theta}$ is the unit vector that defines the direction. The complex conjugate of a complex number $z$ is another complex number $z^*$ such that, if $z = x + iy$, $z^* = x - iy$.

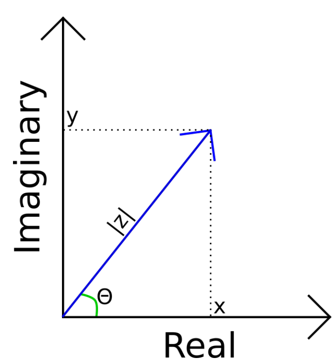


Figure 1: Complex number $z$ represented as a vector in the complex plane.

## Complex Number Properties

**Rectangular vs polar forms:** $z = x + iy = |z|e^{i\theta}$

where $|z| = \sqrt{zz^*} = \sqrt{x^2+y^2}$, $\theta = \text{atan2}(x,y)$. We can also write $x = |z|\cos\theta$, $y = |z|\sin\theta$.

**Euler's identity:** $e^{i\theta} = \cos\theta + i\sin\theta$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

**Complex conjugate:** $z^* = x - iy = |z|e^{-i\theta}$.

$$(z+w)^* = z^* + w^*, (z-w)^* = z^* - w^*$$

$$(zw)^* = z^* w^*, (z/w)^* = z^*/w^*$$

$$z^* = z \Leftrightarrow z \text{ is real}$$

$$(z^n)^* = (z^*)^n$$

## Complex Algebra

Let $z_1 = x_1 + iy_1 = |z_1|e^{i\theta_1}$, $z_2 = x_2 + iy_2 = |z_2|e^{i\theta_2}$.

(Note that we adopt the easier representation between rectangular form and polar form.)

**Addition:** $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

**Multiplication:** $z_1 z_2 = |z_1||z_2|e^{i(\theta_1+\theta_2)}$

**Division:** $\dfrac{z_1}{z_2} = \dfrac{|z_1|}{|z_2|}e^{i(\theta_1-\theta_2)}$

**Power:** $z_1^n = |z_1|^n e^{in\theta_1}$

$z_1^{1/2} = \pm|z_1|^{1/2} e^{i\theta_1/2}$

## Useful Relations

$$-1 = i^2 = e^{i\pi} = e^{-i\pi}$$

$$i = e^{i\pi/2} = \sqrt{-1}$$

$$-i = -e^{i\pi/2} = e^{-i\pi/2}$$

$$\sqrt{i} = (e^{i\pi/2})^{1/2} = \pm e^{i\pi/4} = \frac{\pm(1+i)}{\sqrt{2}}$$

$$-\sqrt{i} = (e^{-i\pi/2})^{1/2} = \pm e^{i\pi/4} = \frac{\pm(1-i)}{\sqrt{2}}$$

**1. Complex Algebra**

(a) Try to express the following values in polar forms: -1, $i$, $-i$, $\sqrt{i}$, and $\sqrt{-i}$.

(b) **Euler's identity.** Represent $\sin\theta$ and $\cos\theta$ using complex numbers.

(c) Show that $|z| = \sqrt{zz^*}$, where $z^*$ is the complex conjugate of $z$.

*Now let's tackle a numerical problem.* Given two complex numbers, $V = 3 - i4$, $I = -(2 + i3)$.

(d) Express $V$ and $I$ in polar form.

(e) Find $VI$, $VI^*$, $V/I$, and $\sqrt{I}$.

(f) What are the roots of $z^2 = 1$? What about $z^3 = 1$? How many roots does $z^n = 1$ have? What is the general form for the solutions of $z^n = 1$?

**2. Binary Addition with Boolean Operators** Here we want to show how to perform addition between two binary numbers.

(a) Compute $(1)_2 + (1)_2$ in binary encoding.

(b) Compute $(1111)_2 + (1001)_2$.

(c) Now let's think about how to express binary addition with Boolean operators. Suppose we want to perform binary addition between two Boolean variables, $A_0$ and $B_0$, we should have two Boolean variables to express the results of carry ($C_0$) and sum ($S_0$). Express the two outputs with Boolean formulae in terms of $A_0$ and $B_0$.

(d) Now we want to perform binary addition between two 2-bit binary numbers, $(A_1A_0)_2$ and $(B_1B_0)_2$. . Could we reuse the results from (c)? *(Hint: could we use $C_0$ to express some formulae?)*

(e) Use the results above to verify $(10)_2 + (11)_2 = (101)_2$.

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