Differential Equations

Order of differential equations

This is the number of the highest derivative in a differential equation. An n^{th} order differential equation is of the form,

$$\alpha_n \frac{\mathrm{d}^n f}{\mathrm{d}t^n}(t) + \alpha_{n-1}\frac{\mathrm{d}^{n-1} f}{\mathrm{d}t^{n-1}}(t) + \cdots + \alpha_1 \frac{\mathrm{d}^1 f}{\mathrm{d}t^1}(t) + \alpha_0 f(t) = 0$$

Where $\alpha_n \neq 0$. The above equation is often denoted in shorthand as,

$$\sum_{k=0}^{n} \alpha_k \frac{\mathrm{d}^k f}{\mathrm{d}x^k}(t) = 0$$

First order differential equations

A first order differential equation is of the form,

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = f(t)$$

Equations of this type have the solution,

$$f(t) = \alpha e^t$$

Here, α is some constant. We can uniquely determine α, and consequently $f(t)$, if we have the evaluation of f at a particular time point t_0. Say $f(t_0) = \alpha_0$. Then,

$$f(t_0) = \alpha e^{t_0} = \alpha_0 \implies \alpha = \alpha_0 e^{-t_0}$$

Thus,

$$f(t) = \left(\alpha_0 e^{-t_0}\right) e^t$$

What about the following differential equation?

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = \lambda f(t)$$

This has the following solution.

$$f(t) = \alpha e^{\lambda t}$$

Again α can be determined from the evaluation of f at a particular time point t_0. We will use this to solve differential equations of higher orders.

Matrix differential equations

It is convenient to represent complicated and correlated differential equations using matrix equations. Let $\vec{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_{n-1}(t) & x_n(t) \end{bmatrix}^T$ be a series of functions of time and let A be the system matrix that describes the differential equation.

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \dots & a_{n,n-1} & a_{n,n} \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} \tag{1}$$

As you can imagine, directly solving this can be a nightmare. This is particularly true if the equations are correlated. For example,

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}(t) = ax_1(t) + bx_2(t) + c\frac{\mathrm{d}x_3}{\mathrm{d}t}(t)$$

We would like to decorrelate our equations in some way. To do so, we consider $\vec{x}(t)$ to be a vector of coordinates, where the first coordinate is $x_1(t)$, the second is $x_2(t)$ and so on. In other words, $\vec{x}(t)$ can be considered to be an element in $\mathbb{R}^n$. We can then decorrelate our equations by changing our coordinates. Assuming A is diagonalizable, let A be decomposed into PDP^{-1}. Here, D is a diagonal matrix with the eigenvalues of A along its diagonal, P is a matrix whose columns are the appropriate eigenvectors of A and P^{-1} is the inverse of P. Then,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) &= A\vec{x}(t) \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) &= PDP^{-1}\vec{x}(t) \\ P^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) &= DP^{-1}\vec{x}(t) \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\left(P^{-1}\vec{x}\right)(t) &= DP^{-1}\vec{x}(t) \end{aligned}$$

Note that we use the fact that A is independent of the time t and that P^{-1} and $\frac{\mathrm{d}}{\mathrm{d}t}(\cdot)$ are linear functions on $\vec{x}(t)$ to conclude that,

$$P^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(P^{-1}\vec{x}\right)(t)$$

By multiplying $\vec{x}(t)$ with P^{-1}, we have changed the coordinate system we are working in into one that decorrelates the data. Let $\vec{y}(t) = P^{-1}\vec{x}(t)$ represent the new coordinate system. Equation (1) becomes,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{n-1}(t) \\ y_n(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{n-1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix}}_{D} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{n-1}(t) \\ y_n(t) \end{bmatrix} \tag{2}$$

Remember, D is a diagonal matrix consisting of the eigenvalues of A along the diagonal. Equation (2) can be succinctly represented as,

$$\frac{\mathrm{d}y_k}{\mathrm{d}t}(t) = \lambda_k y_k(t)$$

This is a first order differential equation that you know how to solve. **Remember to change back into the original coordinate system after solving for $\vec{y}(t)$.**

$$\vec{x}(t) = P\vec{y}(t)$$

Setting up n^{th} order differential equations

You can solve for n^{th} order differential equations using the above matrix set up. Let A describe the relationships within your differential equation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x(t) \\ \frac{\mathrm{d}x}{\mathrm{d}t}(t) \\ \vdots \\ \frac{\mathrm{d}^{n-2}x}{\mathrm{d}t^{n-2}}(t) \\ \frac{\mathrm{d}^{n-1}x}{\mathrm{d}t^{n-1}}(t) \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \dots & a_{n,n-1} & a_{n,n} \end{bmatrix} \begin{bmatrix} x(t) \\ \frac{\mathrm{d}x}{\mathrm{d}t}(t) \\ \vdots \\ \frac{\mathrm{d}^{n-2}x}{\mathrm{d}t^{n-2}}(t) \\ \frac{\mathrm{d}^{n-1}x}{\mathrm{d}t^{n-1}}(t) \end{bmatrix}$$

Example: Second order differential equation

Consider a differential equation of the form,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t) - a_1\frac{\mathrm{d}x}{\mathrm{d}t}(t) - a_0 x(t) = 0$$

This can be written as,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x(t) \\ \frac{\mathrm{d}x}{\mathrm{d}t}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_0 & a_1 \end{bmatrix} \begin{bmatrix} x(t) \\ \frac{\mathrm{d}x}{\mathrm{d}t}(t) \end{bmatrix}$$

Circuits + Differential Equations

The RC circuit is a fundamental component of any real world circuit. Many electronic systems' specifications, like clock speed and bandwidth, are direct results of RC circuits. We will use differential equation

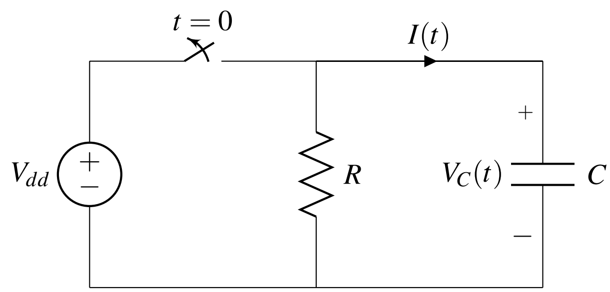

Figure 1: Example Circuit

methods to find the time domain behavior of RC systems. We first set up our problem by defining three functions of time: $I(t)$ is the current at time t, $V(t)$ is the voltage across the circuit at time t, and $V_C(t)$ is the voltage across the capacitor at time t.

Lets consider the RC circuit above in figure 1. Assume the capacitor was fully charged to V_{dd}, and at $t=0$ the switch opens. Now we essentially have a pull-down network bringing the capacitor voltage to 0. After $t=0$ the voltage in the cap is still V_{dd}, starts to flow out of capacitor through the resistor. As the current flows out, the charge stored in the capacitor decreases. This causes the voltage across the capacitor to decrease. How can we describe this behavior mathematically?
Real life is continuous so we need to use differential equations.
Lets start with the relationship between charge and voltage in a capacitor, which is shown below.

$$Q(t) = CV_C(t)$$

We know that there is a current flowing within the capacitor because it is connected to a resistor. Since current is simply the change of the charge over time, we can relate $Q(t)$ to $I(t)$.

$$I(t) = \frac{dQ(t)}{dt}$$

Taking the derivative of the both sides we get

$$\frac{dQ(t)}{dt} = C\frac{dV_C(t)}{dt} = I(t)$$

This is the differential equation that relates the current and voltage of a capacitor.
We know

$$V = IR$$

So we subsitute in for the $I(t)$

$$C\frac{dV(t)}{dt} = -\frac{V_C(t)}{R}$$

We end up with $-\frac{V_C(t)}{R}$ because the current through the resistor is flowing against the direction we defined $I(t)$ as flowing.

Rearranging the term we have

$$\frac{dV}{dt} = -\frac{1}{RC}V$$

$$\frac{d}{dt}V = -\frac{1}{RC}V$$

The differentiation operator is a linear operator, so we can view this equation as a linear system $A*V = \lambda * V$, where λ is the eigenvalue of the system and A is the differentiation operator. Essentially, we need to find a function V that, when operated on by A, results in that same function multiplied by a scale factor. In linear systems, this function is called the eigenfunction of an operator. The eigenfunction of $\frac{d}{dt}$ is $Ke^{\lambda t}$.

When we operate on it with $\frac{d}{dt}$ we get

$$\frac{d}{dt}Ke^{\lambda t} = K\lambda e^{\lambda t} = \lambda K e^{\lambda t}$$

and λ is the corresponding eigenvalue.

In our RC system, the constant K is defined by the boundary conditions and $\lambda = -\frac{1}{RC}$. Plugging our eigenvalue into our eigenfunction gives us

$$V(t) = Ke^{-\frac{t}{RC}}$$

To solve for K, we need to take into account the initial conditions of our problem. At $t = 0$, $V_C(t) = V_{dd}$ so

$$V_C(0) = Ke^{-\frac{0}{RC}} = V_{dd}$$

$$K = V_{dd}$$

Finally we have

$$V_C(t) = V_{dd}e^{-\frac{t}{RC}}$$

Now we can evaluate how long it takes to discharge half the voltage.

$$t_{\text{half life}} = \ln(2)RC$$

The equation is derived by setting $V(t) = \frac{1}{2}$ and solving for t. We see that bigger the values of R and C the longer it takes for the voltage to drop. RC is also called the time constant τ. It's useful to have a general idea of how many τ it takes for a capacitor to reach its final steady state value. After one τ, the capacitor voltage is within 36.8% of its final steady state value. After 5τ, it is within 1% of its final steady state value.

Questions

1. RC Circuits

In this problem, we will be using differential equations to find the voltage across a capacitor V_C over time in an RC circuit. We set up our problem by first defining three functions over time: $I(t)$ is the current at time t, $V(t)$ is the voltage across the circuit at time t, and $V_C(t)$ is the voltage across the capacitor at time t.

Recall from 16A, that the voltage across a resistor is defined as $V_R = RI_R$ where I_R is the current across the resistor. Also, recall that the voltage across a capacitor is defined as $V_C = \frac{Q}{C}$ where Q is the charge across the capacitor.

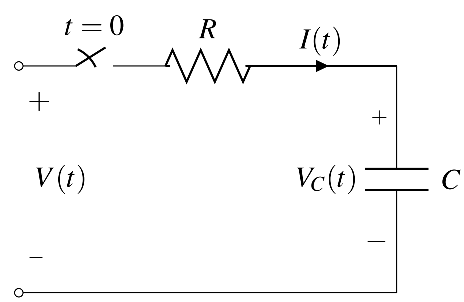

Figure 2: Example Circuit

(a) First, find an equation that relates the current across the capacitor $I(t)$ with the voltage across the capacitor $V_C(t)$.

(b) Using Kirchhoff's law, write an equation that relates the functions $I(t)$, $V_C(t)$, and $V(t)$.

(c) So far, we have three unknown functions and only one equation, but we can remove $I(t)$ from the equation using what we learned in part (a). Rewrite the previous equation in part (b) in the form of a differential equation.

(d) Let's suppose that for $t < 0$ the capacitor is precharged to a voltage V_{DD} and that $V(t) = 0\ \forall t \geq 0$, simply a short to ground. Assuming that we close the switch at $t = 0$, use the fact that $V_C(0) = V_{DD}$ to solve this differential equation for $V_C(t)$.

(e) Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit. Assuming the switch closes at $t = 0$, use your differential equation to solve for $V_C(t)$.

(f) Now that you know how the voltage across a capacitor acts over time in an RC circuit, how does the charge in the capacitor act over time? Write your answer as a function of $Q(t)$, and remember that $V_C = \frac{Q}{C}$.

2. Two Capacitors

Consider the circuit below, assume that when $t \leq 0$, both capacitors have no charge ($V_1(t=0) = 0$ and $V_2(t=0) = 0$). At $t = 0$, the switch closes.

(a) First, use Kirchoff's Laws and the capacitor equation ($I = \frac{dV}{dt}C$) to find the differential equation of this circuit.

(b) As shown in class, we can write each voltage $V(t)$ as $V_{std}(t) + V_{trans}(t)$, where $V_{std}(t)$ is the steady state function and $V_{trans}(t)$ is the transient function.

What are the steady state functions of $V_1(t)$ and $V_2(t)$?

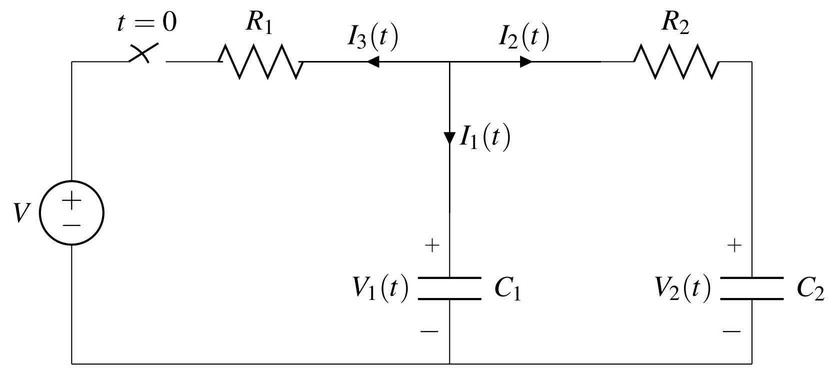

Figure 3: Two Capacitor Circuit with Voltage Source

(c) Now, replace $V_1(t) = V_{1,std}(t) + V_{1,trans}(t)$ and $V_2(t) = V_{2,std}(t) + V_{2,trans}(t)$ using the steady state functions you found in the previous part. Also, find the initial conditions of $V_{1,trans}(t)$ and $V_{2,trans}(t)$ when $t = 0$.

(d) Assume that $C_1 = C_2 = 1$, $R_1 = \frac{1}{3}$, and $R_2 = \frac{1}{2}$. Diagonalize the matrix A in $\frac{d}{dt}(\begin{bmatrix} V_{1,trans}(t) \\ V_{2,trans}(t) \end{bmatrix}) = A \begin{bmatrix} V_{1,trans}(t) \\ V_{2,trans}(t) \end{bmatrix}$

(e) Now that we have diagonalized the matrix, we can now work in the eigenspace. Let us call the transformed $\begin{bmatrix} V_{1,trans}(t) \\ V_{2,trans}(t) \end{bmatrix}$ as $\tilde{V}(t) = \begin{bmatrix} \tilde{V_1}(t) \\ \tilde{V_2}(t) \end{bmatrix}$. Solve for $\tilde{V}(t)$. Do not forget about the initial condtions of $\tilde{V}(t)$.

(f) Now that we have $\tilde{V}(t)$, find the solution for $\begin{bmatrix} V_{1,trans}(t) \\ V_{2,trans}(t) \end{bmatrix}$.

(g) For the final step, solve for $\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$.

(h) Sketch the voltage vs time plots of $V_1(t)$ and $V_2(t)$.

Contributors:

- Siddharth Iyer.
- Saavan Patel, Brian Kilberg.