

Transfer Functions

When we analysed circuits in the phasor domain, we always told you what the input voltage, or the input sinusoid to the circuit was. However, sometimes we have many input sinusoids, and we want to look at how a circuit (or system) generically responds to a sinusoid input of frequency ω . We want to see how an input sinusoid "transfers" into an output sinusoid. How do we do this?

Let's start with the example of an RC Circuit.

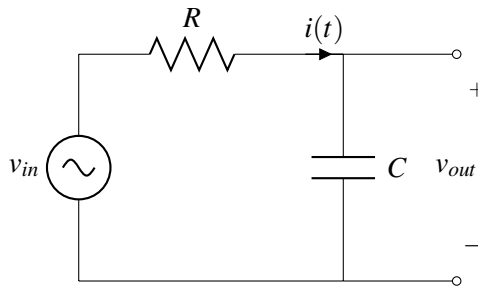


Figure 1: A first order RC Low Pass Filter

In the phasor domain, the impedance of the capacitor is $Z_C = \frac{1}{j\omega C}$ and the impedance of the Resistor is $Z_R = R$. Because we treat impedances the same as resistances, this circuit looks like a voltage divider in the phasor domain.

$$v_{out} = \frac{Z_C}{Z_R + Z_C} v_{in} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} v_{in} = \frac{1}{j\omega RC + 1} v_{in}$$

We define the frequency response as

$$H(\omega) = \frac{v_{out}}{v_{in}} = \frac{1}{j\omega RC + 1}$$

Now, given an arbitrary input sinusoid if we multiply it by the frequency response, we can get the output sinusoid. What this allows us to do, is model any arbitrary circuit as a 4 port (two input, two output) black box. The transfer function completely defines how our circuit works.

The transfer function of a circuit provides deeper insight than the DC response of a circuit. Let's consider the transfer function above. How does the circuit respond to high frequencies? What about low frequencies?

$$\lim_{\omega \rightarrow \infty} H(\omega) = \frac{1}{j\infty RC + 1} = 0$$

$$\lim_{\omega \rightarrow 0} H(\omega) = \frac{1}{j0RC + 1} = 1$$

This tells us that the RC circuit above passes sinusoids of lower frequencies (frequencies close to 0), and stops sinusoids of high frequencies (frequencies closer to ∞). For this reason we call it a "low pass" filter. If we input a combination of sinusoids of different frequencies into this filter, we can see that it will respond differently to all of them.

When we write the transfer function of an arbitrary circuit, it always takes the following form. This is called a "rational transfer function". We also like to factor the numerator and denominator, so that they become easier to work with and plot:

$$H(\omega) = \frac{n(\omega)}{d(\omega)} = \frac{(j\omega)^n \alpha_n + (j\omega)^{n-1} \alpha_{n-1} + \dots + j\omega \alpha_1 + \alpha_0}{(j\omega)^n \beta_n + (j\omega)^{n-1} \beta_{n-1} + \dots + j\omega \beta_1 + \beta_0} = K \frac{(j\frac{\omega}{\omega_{z1}} + 1)(j\frac{\omega}{\omega_{z2}} + 1)\dots}{(j\frac{\omega}{\omega_{p1}} + 1)(j\frac{\omega}{\omega_{p2}} + 1)\dots}$$

Here, we define the constants ω_z as "zeros" and ω_p as "poles". In the circuit above, we have one no zeros and 1 pole at $\omega_p = \frac{1}{RC}$. The significance of poles and zeros will be explained in further detail when we go into Bode Plots.

Note: If we cascade transfer circuits to make more complex transfer functions, we can't treat the transfer functions separately. To illustrate this, let's take the example above and cascade it.

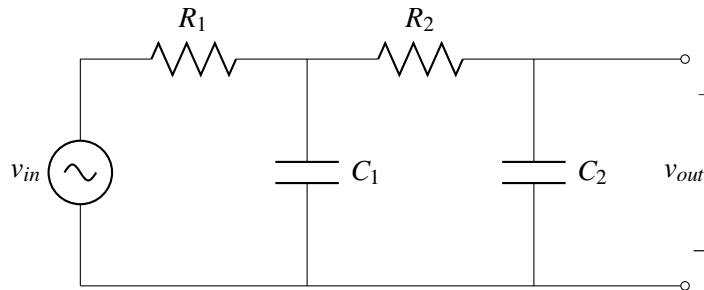


Figure 2: Second Order RC Filter

If we set the intermediate node to V_x , we get the following equations to solve for the transfer function.

Performing KCL at the V_x node and the V_{out} node.

$$\frac{V_x - V_{in}}{R_1} + \frac{V_x}{\frac{1}{j\omega C_1}} + \frac{V_x - V_{out}}{R_2} = 0$$

$$\frac{V_{out} - V_x}{R_2} + \frac{V_{out}}{\frac{1}{j\omega C_2}} = 0$$

Putting the equations together and solving for the transfer function, we get:

$$H(\omega)_{RCActual} = \frac{V_{out}}{V_{in}} = \frac{1}{(j\omega)^2 R_1 R_2 C_1 C_2 + j\omega(C_2 R_1 + C_1 R_1 + R_2 C_2) + 1} = \frac{1}{(j\omega R_1 C_1 + 1)(j\omega R_2 C_2 + 1) + j\omega C_2 R_1}$$

If we had thought of the circuit in Figure 2 as two separate RC Low Pass filters (as shown in Figure 1), then our frequency response would have been

$$H(\omega)_{RCPredicted} = \frac{1}{(j\omega R_1 C_1 + 1)(j\omega R_2 C_2 + 1)}$$

The predicted solution that we have is incorrect for the above circuit (Figure 2). Because there is coupling between the two RC filters, we can't model it as two separate RC filters. If we wanted to do that, we would have to find a way to de-couple the two circuits from each other. Luckily, we learned how to do this! If we put an op-amp in buffer configuration between the two RC filters, they are no longer electrically coupled to one another, and act as two separate RC low pass filters.

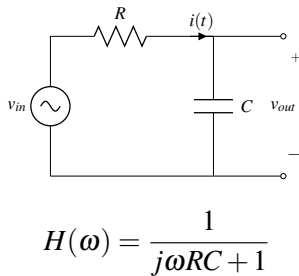


Figure 3: Simple RC Low Pass Filter

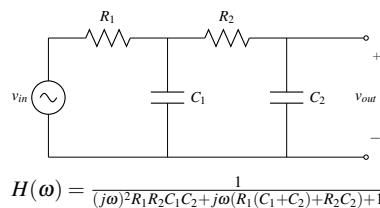


Figure 4: Cascaded RC Filter with no buffer stage

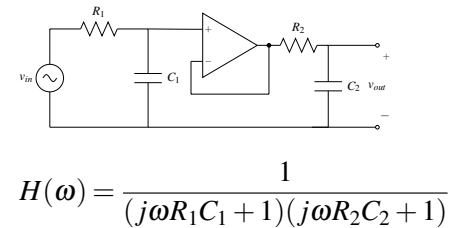


Figure 5: Cascaded RC Filter with intermediate buffer stage

Plotting Transfer Functions

It is often useful to be able to plot the transfer function of a circuit as a function of frequency. This allows to visualize how our circuit reacts to any particular frequency. Because the frequency response is a complex number, we plot the magnitude and phase of the frequency response on separate plots. Transfer functions can be very complicated, so being able to plot them allows to look at how the circuit responds at intermediate frequencies, not just the extremes of 0 and ∞ .

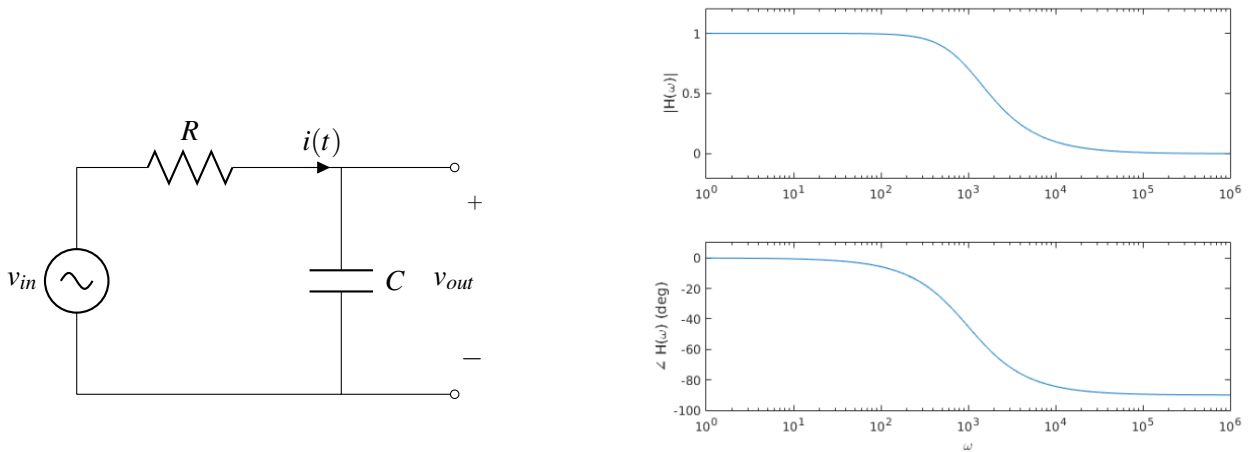
For example, let's plot the transfer function for the RC Low pass filter above. First, we find the magnitude and phase response from the transfer function.

$$|H(\omega)| = \left| \frac{1}{j\omega RC + 1} \right| = \frac{|1|}{|j\omega RC + 1|} = \frac{1}{\sqrt{(\omega RC)^2 + 1}}$$

$$\angle H(\omega) = \angle \frac{1}{j\omega RC + 1} = \angle 1 - \angle(j\omega RC + 1) = -\tan^{-1}(\omega RC)$$

Plotting using numerical methods and $R = 1k\Omega, C = 1\mu F$ we get the following:

There a few things to notice about the plot. We plotted the frequency on a logarithmic scale so that we can see the circuit response over a wide range. One important thing to notice is that at the pole frequency ($\omega_p = \frac{1}{RC} = 1000 \frac{rad}{s}$), the phase is precisely -45° and the magnitude is precisely $\frac{1}{\sqrt{2}} \approx 0.707$. We refer to this point as the "cutoff frequency" for this low pass filter. Every frequency below the cutoff frequency is "passed" (its amplitude remains the same), and everything above the cutoff frequency is stopped (its amplitude goes to ≈ 0). This is an approximation, but as long as we design our circuit to have minimal frequencies near the cutoff, it is a pretty good one.



$$H(\omega) = \frac{1}{j\omega RC + 1} \quad |H(\omega)| = \frac{1}{\sqrt{(\omega RC)^2 + 1}} \quad \angle H(\omega) = -\tan^{-1}(\omega RC)$$

$$R = 1k\Omega, C = 1\mu F$$

$$\omega_p = \frac{1}{RC} = 1000 \frac{rad}{s}$$

Figure 6: A first order RC Low Pass Filter

Bode Plots

Bode Plots provide us with a simple and easy tool to plot these transfer functions by hand. Always remember that Bode Plots are an approximation, if you want the precisely correct plots you need to use numerical methods (like solving using MATLAB or ipython).

When we make Bode Plots we plot the frequency on a logarithmic scale, the magnitude on a decibel scale and the angle in either degrees or radians. We use the decibel because it allows us to break up complex transfer functions into it's constituent components. We define the decibel as the following:

$$20\log_{10}(|H(\omega)|) = dB$$

We know that a transfer function for a circuit takes the following general form:

$$\begin{aligned} H(\omega) &= \frac{(j\frac{\omega}{\omega_{z1}} + 1)(j\frac{\omega}{\omega_{z2}} + 1)\dots}{(j\frac{\omega}{\omega_{p1}} + 1)(j\frac{\omega}{\omega_{p2}} + 1)\dots} \\ \log(H(\omega)) &= \log\left(\frac{(j\frac{\omega}{\omega_{z1}} + 1)(j\frac{\omega}{\omega_{z2}} + 1)\dots}{(j\frac{\omega}{\omega_{p1}} + 1)(j\frac{\omega}{\omega_{p2}} + 1)\dots}\right) \\ &= \log\left(\left(j\frac{\omega}{\omega_{z1}} + 1\right)\left(j\frac{\omega}{\omega_{z2}} + 1\right)\dots\right) - \log\left(\left(j\frac{\omega}{\omega_{p1}} + 1\right)\left(j\frac{\omega}{\omega_{p2}} + 1\right)\dots\right) \\ &= \log\left(j\frac{\omega}{\omega_{z1}} + 1\right) + \log\left(j\frac{\omega}{\omega_{z2}} + 1\right) + \dots - \log\left(j\frac{\omega}{\omega_{p1}} + 1\right) - \log\left(j\frac{\omega}{\omega_{p2}} + 1\right) - \dots \\ &= \sum \log\left(j\frac{\omega}{\omega_z} + 1\right) - \sum \log\left(j\frac{\omega}{\omega_p} + 1\right) \end{aligned}$$

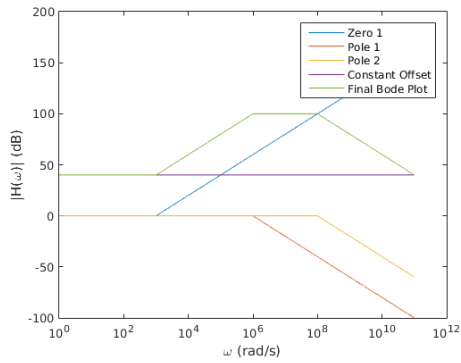


Figure 7: Plotting of the transfer function magnitude using the Bode approximation

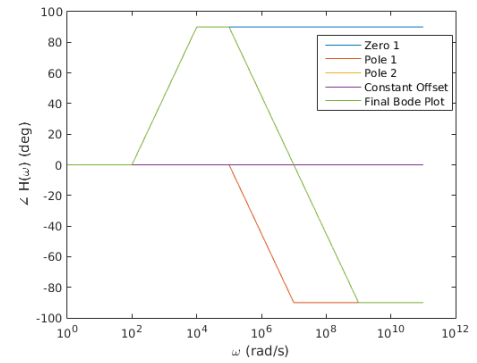


Figure 8: Plotting of the transfer function angle using the Bode approximation

This means that when making the bode plot (and plotting using a logarithmic unit), we can treat each individual pole and zero independently, and then add them back together at the end. We can use the Bode Plot rules (shown on the next page) to help us plot each of the individual poles and zeros.

Aside: Where did the factor 2 come from in the conversion between Voltage and Decibels? Because current also matters to us when looking at these transfer functions, a better metric for transfer function magnitude is power transfer instead of voltage transfer.

$$|H(\omega)| = 10 \log \frac{P_{out}}{P_{in}} = 10 \log \frac{V_{out}^2}{R} = 10 \log \frac{V_{out}^2}{V_{in}^2} = 20 \log \frac{V_{out}}{V_{in}}$$

Example:

Plot the magnitude and phase of the following transfer function using the Bode approximation and a numerical solver and compare the two.

$$H(\omega) = \frac{100(j\frac{\omega}{1000} + 1)}{(j\frac{\omega}{10^6} + 1)(j\frac{\omega}{10^8} + 1)}$$

Solution:

We see 1 zero at $10^3 \frac{rad}{s}$, 2 poles at $10^6 \frac{rad}{s}$ and $10^8 \frac{rad}{s}$, and a constant offset by 100. We start with the constant value, and then move from lowest to highest frequency plotting the poles and zeros as we go. Finally, we add together the plots for each of the individual poles and zeros to give us the final Bode Plot.

Finally, comparing the Bode approximation and the precise value calculated via a computer, we can see the Bode approximation is very similar to the exact answer, except for around the pole and zero frequencies (as expected).

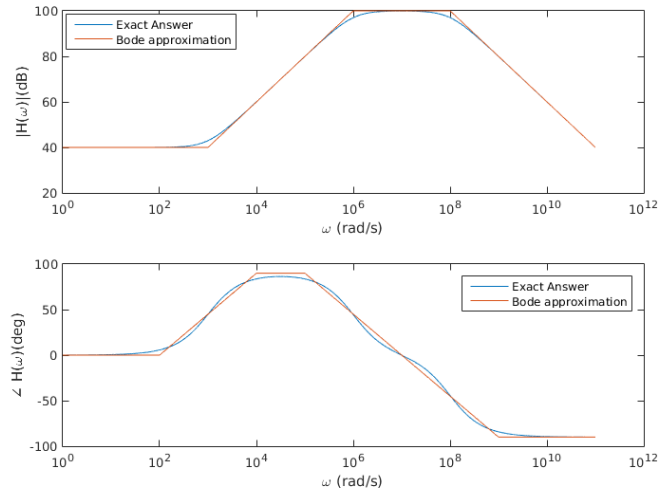


Figure 9: A comparison of Bode vs. Exact (numerically computed) answers. Note the good agreement between both, except at the pole and zero frequencies.

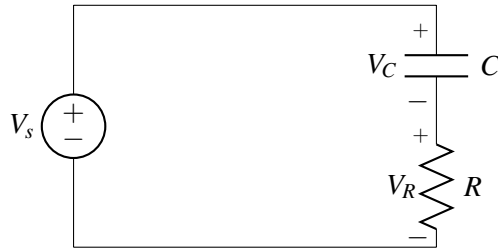
Bode Plots

Factor	Bode Magnitude	Bode Phase
Constant K	$20 \log K$ 0 dB	$\pm 180^\circ$ if $K < 0$ 0° if $K > 0$
Zero @ Origin $(j\omega)^N$	slope = $20N$ dB/decade 0 dB	$(90N)^\circ$ 0°
Pole @ Origin $(j\omega)^{-N}$	slope = $-20N$ dB/decade 0 dB	$(-90N)^\circ$ 0°
Simple Zero $(1 + j\omega/\omega_c)^N$	slope = $20N$ dB/decade ω_c	$(90N)^\circ$ 0° $0.1\omega_c$ ω_c $10\omega_c$
Simple Pole $\left(\frac{1}{1 + j\omega/\omega_c}\right)^N$	slope = $-20N$ dB/decade ω_c	$(-90N)^\circ$ 0° $0.1\omega_c$ ω_c $10\omega_c$
Quadratic Zero $[1 + j2\xi\omega/\omega_c + (j\omega/\omega_c)^2]^N$	slope = $40N$ dB/decade ω_c	$(180N)^\circ$ 0° $0.1\omega_c$ ω_c $10\omega_c$
Quadratic Pole $\frac{1}{[1 + j2\xi\omega/\omega_c + (j\omega/\omega_c)^2]^N}$	slope = $-40N$ dB/decade ω_c	$(-180N)^\circ$ 0° $0.1\omega_c$ ω_c $10\omega_c$

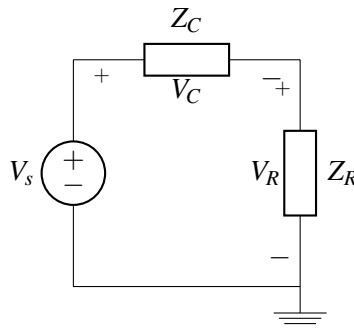
Questions

1. Bode Plots of Transfer Functions

To understand the concept of transfer functions and filters with a concrete example, consider the following simple RC circuit. Let the voltage source V_S be designated as the input phasor, and let V_R and V_C designate the two output voltage phasors. $R = 1\text{k}\Omega$ and $C = 1\mu\text{F}$.



- What is the impedance of a $1\text{k}\Omega$ resistor? Draw a bode plot of the impedance of the resistor as a function of frequency.
- What is the impedance of a $1\mu\text{F}$ capacitor? On the same bode plot as the last question, sketch the capacitor's impedance as a function of frequency. When is $|Z_C| \gg |Z_R|$ and vice versa? At what ω does $|Z_C| = |Z_R|$? What is this ω called?
- Now let's look at the impedance voltage divider below. What is the transfer function $H(\omega) = \frac{V_C}{V_S}$?



- For the region where $|Z_R| \gg |Z_C|$, what is the approximate function for $|\frac{V_C}{V_S}|$? Sketch a bode plot of this function for this region. At what frequencies is our approximation no longer valid?
- What is the approximate function for $|\frac{V_C}{V_S}|$ when $|Z_C| \gg |Z_R|$? On the same plot as before, sketch the bode plot of this approximate function. Where does this function meet your approximation for when $|Z_R| \gg |Z_C|$?
- What is the worst case error for our piecewise approximation? On a log-log plot, is this error appear very large?
- Approximately what is the phase of $\frac{V_C}{V_S}$ at $\omega = 0, \frac{1}{10RC}, \frac{1}{RC}, \frac{10}{RC}, \frac{1000}{RC}$? Connect the dots and sketch a plot of phase vs. time below your magnitude plot.
- Draw the bode plot for $|\frac{V_R}{V_S}|$.

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