EECS 16B Designing Information Devices and Systems II
Fall 2016 Murat Arcak and Michel Maharbiz **Discussion 4B**

Change of basis review

In an attempt to make the concept basis, and the change of basis, more intuitive, we are going to exploit the interpretation that basis describe the structure of the space that we are working in. Consider the following two bases.

$$\mathscr{U} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\} \text{ and } \mathscr{V} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$

Let $\vec{x}$ be any arbitrary vector. In order to describe $\vec{x}$ in terms of $\mathscr{U}$ and $\mathscr{V}$, we use the *coordinates* of $\vec{x}$ with respect to $\mathscr{U}$ and $\mathscr{V}$ respectively. Let $\{\alpha_k\}_{k=1}^n$ and $\{\beta_k\}_{k=1}^n$ be the coordinates of $\vec{x}$ with respect to $\mathscr{U}$ and $\mathscr{V}$ respectively.

$$\vec{x} = \sum_{k=1}^{n} \alpha_k \vec{u}_k \text{ in terms of } \mathscr{U}$$
$$\vec{x} = \sum_{k=1}^{n} \beta_k \vec{v}_k \text{ in terms of } \mathscr{V}$$

Since these are alternate representations of the same vector, it follows that,

$$\sum_{k=1}^{n} \alpha_k \vec{u}_k = \sum_{k=1}^{n} \beta_k \vec{v}_k$$

In matrix form, the above can be represented as,

$$\underbrace{\begin{bmatrix} | & & | \\ \vec{u}_1 & \ldots & \vec{u}_n \\ | & & | \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\vec{\alpha}} = \underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_n \\ | & & | \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}}_{\vec{\beta}}$$

or,

$$U\vec{\alpha} = V\vec{\beta}$$

If we are working in the space $\mathscr{U}$, $\vec{x}$ is represented by $\vec{\alpha}$. To change coordinates, or the working space, to the space $\mathscr{V}$, we note that,

$$\vec{\beta} = V^{-1}U\vec{\alpha}$$

We essentially pipe $\vec{\alpha}$ through $V^{-1}U$ by left multiplying $\vec{\alpha}$ with $V^{-1}U$ to obtain $\vec{\beta}$, which is $\vec{x}$ in the space $\mathscr{V}$.

Matrix representation review

In this section, we will describe what a matrix's construction implies. Consider the following two basis.

$$\mathscr{U} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\} \text{ and } \mathscr{V} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$

Let matrix A be a matrix whose domain is the space spanned by $\mathscr{U}$ and co-domain is $\mathscr{V}$. Consider the k^{th} column of A.

$$A_k = \begin{bmatrix} a_{1,k} \\ a_{2,k} \\ \vdots \\ a_{n,k} \end{bmatrix}$$

This columns tells us how A acts on the k^{th} basis vector of the domain $\mathscr{U}$.

$$A\vec{u}_k = a_{1,k}\vec{v}_1 + a_{2,k}\vec{v}_2 + \cdots + a_{n,k}\vec{v}_n$$

For example, consider a matrix A from vector space $\mathscr{E}$ to vector space $\mathscr{E}$, where,

$$\mathscr{E} = \{\vec{e}_1, \vec{e}_2\}$$

such that,

$$A\vec{e}_1 = \vec{e}_1 \text{ and } A\vec{e}_2 = \vec{e}_1 + \vec{e}_2$$

This tells us all we need to know in terms of constructing A as a matrix that acts from $\mathscr{E}$ to $\mathscr{E}$.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Discrete Time State Space Models

Discrete time state space models allow us to model the progression of discrete time systems, similar to how matrix differential equations allowed us to model how continuous time systems evolved with time. In general, you can transform between continuous and discrete time models using a process called "linearization", which will be covered later in the course. Because computers work in discrete time rather than continuous time, it is often more useful to work with discrete time systems than it is to work with continuous time systems.

$$\frac{d}{dt}\vec{x}(t) = A_{ct}\vec{x}(t)$$

Figure 1: Continuous time state space model (written as differential equation)

$$\vec{x}[n+1] = A_{dt}\vec{x}[n]$$

Figure 2: Discrete time state space model (written as recursive equation)

When we are using these equations to model how a system evolves with time, we also like to add an input vector that represents our modification to the system with time. Here, A is an $n \times n$ matrix, $\vec{x}[n]$ is a $nx1$

vector, b is an nxm matrix and $\vec{u}[n]$ is a $mx1$ vector. In our models, A and b do not vary with time, but $\vec{u}[n]$ and $\vec{x}[n]$ do.

$$\vec{x}[n+1] = A\vec{x}[n] + b\vec{u}[n]$$

Let's consider the "no-input" case, where $\vec{u}[n] = 0$. How does our system evolve with time? To examine this problem, we need to move the system to the eigenbasis, and see how the individual components evolve with time.

$$\begin{aligned}
\vec{x}[n+1] &= A\vec{x}[n] \\
&= PDP^{-1}\vec{x}[n] \\
P^{-1}\vec{x}[n+1] &= DP^{-1}\vec{x}[n] \\
\tilde{\vec{x}}[n+1] &= D\tilde{\vec{x}}[n]
\end{aligned}$$

By moving into the eigenbasis, we have decoupled each of the individual components of $\tilde{x}[n]$, allowing us to treat each of them individually.

$$\tilde{\vec{x}}[n+1] = \begin{bmatrix} \tilde{x_1} \\ \tilde{x_2} \\ \tilde{x_3} \\ \vdots \\ \tilde{x_m} \end{bmatrix} [n+1] = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \lambda_m \end{bmatrix} \begin{bmatrix} \tilde{x_1} \\ \tilde{x_2} \\ \tilde{x_3} \\ \vdots \\ \tilde{x_m} \end{bmatrix} [n]$$

$$\tilde{x_k}[n+1] = \lambda_k \tilde{x_k}[n]$$

This is a similar relation to the one we had seen in the decoupled basis for differential equations, but this is a recurrence relation, instead of an eigenvalue/eigenfunction problem. We can see this by the following:

$$\begin{aligned}
\tilde{x_k}[1] &= \lambda_k \tilde{x_k}[0] \\
\tilde{x_k}[2] &= \lambda_k \tilde{x_k}[1] = \lambda_k^2 \tilde{x_k}[0] \\
&\vdots \\
\tilde{x_k}[n+1] &= \lambda_k \tilde{x_k}[n] \\
&= (\lambda_k)^{n+1} \tilde{x_k}[0]
\end{aligned}$$

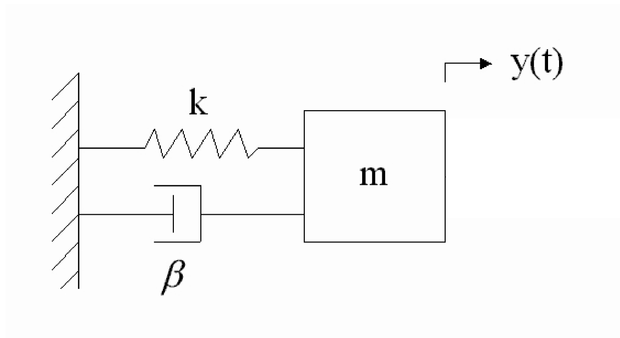
By transforming the eigenbasis ($\tilde{\vec{x}}$) to the regular basis ($\vec{x}$) we can see how our system evolves with time. We will be going into more detail about the implications of this in further discussions and notes.

Questions

1. Harmonic Motion

Consider the mass-spring-damper system below, described by

$$m\ddot{y} + \beta\dot{y} + ky = 0.$$



(a) Choose two states x_1 and x_2 and write this system in state space form $\dot{x} = Ax$.

(b) Compute the eigenvalues of the matrix A when

i. $m = 1$, $k = 1$, $\beta = 0$

ii. $m = 1$, $k = 1$, $\beta = 1$.

iii. $m = 1$, $k = 1$, $\beta = 2$.

What behavior do you expect from the system in each of these cases?

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