Controllability

We are given a continuous time state space system, where $\vec{x}$ is our state vector, A is the state space model, B is the input matrix and $\vec{u}$ is the control input.

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

We want to know if this system is "controllable"; if a given set of inputs we can get the system from any initial state to any final state. This has important physical meaning; if a physical system is controllable, that means we can get anywhere in the state space. If a robot is controllable it is able to travel anywhere in the system it is living in (given enough control inputs).

Constructing the Controllability matrix

To figure out if a system is controllable, we can simplify the problem. If we want to reach any final state from any initial state, we can consider the initial state as the origin and the final state as any arbitrary point in the state space. A system is controllable if we start off at $x\vec{(t)} = 0$ and after some set of control inputs $u(t)$ we can reach an arbitrary final state $x\vec{(t)} = x_0$. Let's start the system off at $x\vec{(t)} = 0$ and see how the system evolves with each time step.

$$\dot{\vec{x}}(t) = B\vec{u}(t)$$

This shows us that we can go anywhere spanned by B in the first time step. Using our input vector u, we can push the system anywhere the matrix B lets us go. Now for the next time step. The time derivative of the function defines where it will be at the next time step, lets call this $\vec{x}(t+\Delta t)$.

$$\begin{aligned}\dot{\vec{x}}(t+\Delta t) &= A\vec{x}(t+\Delta t) + B\vec{u}(t+\Delta t)\\ &= AB\vec{u}(t) + B\vec{u}(t+\Delta t)\end{aligned}$$

Similarly, at this time step, we can go anywhere spanned by $\begin{bmatrix} B & AB \end{bmatrix}$. Every time step adds another degree of freedom to the system.

If we go another time step, $\vec{x}(t+2\Delta t)$, we get the following

$$\begin{aligned}\dot{\vec{x}}(t+2\Delta t) &= A\vec{x}(t+2\Delta t) + B\vec{u}(t+2\Delta t)\\ &= A^2B\vec{u}(t) + AB\vec{u}(t+\Delta t) + B\vec{u}(t+2\Delta t)\end{aligned}$$

After k time steps, we get the following

$$\begin{aligned}\dot{\vec{x}}(t+k\Delta t) &= A\vec{x}(t+k\Delta t)+B\vec{(}t+k\Delta t)\\ &= A^kB\vec{u}(t)+A^{k-1}B\vec{u}(t+\Delta t)+A^{k-2}B\vec{u}(t+2\Delta t)+...AB\vec{u}(t+(k-1)\Delta t)+B\vec{u}(t+k\Delta t)\end{aligned}$$

After 1 time step, we can go anywhere in the set of vectors spanned by B, after 2 time steps, we can go anywhere spanned by $\begin{bmatrix}B & AB\end{bmatrix}$, and after k time steps, we can go anywhere spanned by the set C, defined below. This is what is called the "controllability" matrix.

$$C = \begin{bmatrix}B & AB & A^2B & ... & A^{k-1}B & A^kB\end{bmatrix}$$

If this matrix is of rank n (the dimension of our state space), then we know that our system is fully controllable. It means that our control system forms a surjective map onto the state space. But what if we didn't take enough sets, and the last vector needed to span the system is at time step $k+1$? What is the minimal number of steps we need to take to ensure that we have a set of control inputs that fully span the set?

Cayley-Hamilton

These questions are answered by the Cayley-Hamilton theorem. The Cayley-Hamilton theorem says that higher order powers of A can be expressed as a linear combination of lower order matrix powers of A. Specifically if A is an $n \times n$, matrix, the highest order unique power of A is A^{n-1}. Thus, if we keep applying control inputs past n time steps, our control inputs will be a linear combination of the previous control inputs, and cannot increase the rank of the controllability matrix.

Definition of Controllability

This also works for discrete time systems, instead of incrementing our system by time steps Δt, we increment by discrete time intervals $t+1, t+2$ etc. The math, and our controllability test works out to be exactly the same! Putting all of this together we get the following:

$$\dot{\vec{x}}(t) = A\vec{x}(t)+B\vec{u}(t), \quad \vec{x}[t+1] = A\vec{x}[t]+B\vec{u}[t]$$
$$C = \begin{bmatrix}B & AB & A^2B & ... & A^{n-2}B & A^{n-1}B\end{bmatrix}$$

Given a continuous or discrete time system x with dimensionality n. The system is controllable if it's controllability matrix C is of rank n. If a system is controllable, then given a starting position $\vec{x}(t) = 0$, it takes a maximum n control inputs over n time steps for the system to reach any final state $\vec{x}(t) = \vec{x}_0$.

Questions

1. **Deadbeat Control**

   Consider the system

$$x(t+1) = Ax(t)+Bu(t) = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}x(t)+\begin{bmatrix}0\\ 1\end{bmatrix}u(t).$$

   (a) Is this system controllable?

(b) For which initial states $x(0)$ is there a control that will bring the state to zero in a single time step?

(c) For which initial states $x(0)$ is there a control that will bring the state to zero in two time steps?

2. Cayley and Hamilton

Cayley is trying to control the system

$$x(t+1) = Ax(t) + Bu(t) = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

using feedback.

(a) Is this system stable?

(b) Is this system controllable?

(c) Cayley has been computing for a while trying to find some k so that the matrix

$$\mathscr{C}_k = \begin{bmatrix} B & AB & A^2B & \dots & A^kB \end{bmatrix}$$

has rank 2, but still hasn't found one. Confirm that for $k = 3$ this matrix still has rank 1.

(d) Cayley's friend Hamilton remembers hearing somewhere that for any $n \times n$ matrix A, the matrix A^n can always be written as a linear combination of A^{n-1}, A^{n-2}, $\dots$, A and I.[1] Is this true for the A matrix of Cayley's system?

(e) Will Cayley ever find some k to make

$$\mathscr{C}_k = \begin{bmatrix} B & AB & A^2 & \dots & A^kB \end{bmatrix}$$

have rank 2?

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[1] Hamilton is right about this. It follows from a result known at the Cayley-Hamilton Theorem which says that any $n \times n$ matrix always satisfies its characteristic equation. So the characteristic equation $\lambda^2 - 2\lambda - 8$ we derived above implies that $A^2 - 2A - 8 = 0$. You'll learn more about this theorem if you take the advanced control course EE 221A.