Observers and Observability

So far we have always assumed that we are able to directly see the A matrix, and we know the output of the system perfectly at all times. This isn't always the case, as the equations governing a given system are not always visible at the output (think of a robot car, we can see how the wheels move, but we are unable to see the voltages and currents in the motor actually governing the motion of the car).

$$\begin{aligned}\vec{x}(t+1) &= A\vec{x}(t) + B\vec{u}(t)\\ \vec{y}(t) &= C\vec{x}(t)\,.\end{aligned}$$

In this system the $\vec{y}$ represents the outputs that we are able to see and measure when looking at the system. We are usually not able to directly observe the state space system of the matrix, or the contents of the A matrix. We define the concept of *Observability* as the ability to determine the initial state $\vec{x}(0)$ from a finite series of measurements. Using a similar argument as controllability, the observability matrix, O can be constructed the following way.

Constructing the Observability matrix

At time step zero we are able to see the output $\vec{y}(0) = C\vec{x}(0)$. If we let the system evolve with time without any control input, we can reconstruct the A matrix.

$$\begin{aligned}\vec{y}(0) &= C\vec{x}(0)\\ \vec{y}(1) &= C\vec{x}(1) = CA\vec{x}(0)\\ \vec{y}(2) &= CA^2\vec{x}(0)\\ &\vdots\\ \vec{y}(k) &= CA^k\vec{x}(0)\end{aligned}$$

Rewriting this, we get the following relation

$$\begin{bmatrix}\vec{y}(0)\\ \vec{y}(1)\\ \vec{y}(2)\\ \vec{y}(3)\\ \vdots\\ \vec{y}(k)\end{bmatrix} = \begin{bmatrix}C\\ AC\\ A^2C\\ A^3C\\ \vdots\\ A^kC\end{bmatrix}\vec{x}(0)$$

If we want to be able to uniquely determine the original input $\vec{x}(0)$, we want each of the rows of this matrix to be linearly independent. By the Cayley-Hamilton theorem, we know that the maximum linearly independent power of A^kC is the one where $k = n-1$ where n is the dimension of our state space. Thus, we define the observability matrix O to be the following.

$$O = \begin{bmatrix} C \\ AC \\ A^2C \\ A^3C \\ \vdots \\ A^{n-1}C \end{bmatrix}$$

If the O matrix is full rank, then the system is observable, and we can uniquely determine the initial state from a series of inputs. Notice the similarity to the Controllability matrix. In fact, we call observability the mathematical dual of controllability.

Observers

So far, we have neglected to mention how our state-feedback derive the information about the state, when the state is not directly measurable.

We need to construct an additional system that provides an estimate $\vec{\hat{x}}$ of the state $\vec{x}$ using the inputs $\vec{u}$ and the outputs $\vec{y}$. Given the system

$$\begin{aligned} \vec{x}(t+1) &= A\vec{x}(t) + B\vec{u}(t) \\ \vec{y}(t) &= C\vec{x}(t)\,. \end{aligned}$$

We construct an additional system, called an *observer*, to estimate the state:

$$\begin{aligned} \vec{\hat{x}}(t+1) &= A\vec{\hat{x}}(t) + B\vec{u}(t) - \underbrace{L\left(\vec{\hat{y}}(t) - \vec{y}(t)\right)}_{\text{output feedback}} \\ \vec{\hat{y}}(t) &= C\vec{\hat{x}}(t)\,. \end{aligned}$$

The additional output-feedback term tracks the difference between the estimated and real outputs. If the observer outputs deviate from the real outputs, it nudges the observer closer to the real outputs. If L is chosen the right way, the estimate quickly converges to the real state.

Subtracting the system from its observer, we can find a difference equation describing the error $\vec{e}(t) = \vec{\hat{x}}(t) - \vec{x}(t)$:

$$\vec{e}(t+1) = (A - LC)\,\vec{e}(t)$$

This equation is similar to state feedback, but the order of LC vs. BK is different. To handle this difficulty we introduce the **dual system**. It is a formal system—we do not care about its behavior—we just use it to find L.

$$\vec{z}(t+1) = A^T\vec{z}(t) + C^T\vec{v}(t)$$

This system is controllable if the original system is observable (the controllability matrix of the dual system is the transpose of the observability matrix of the original system).

We now find the feedback L^T that places the eigenvalues of $A^T - C^T L^T$ where we want them to be. Then transpose the result and get $A - LC$ with the the same eigenvalues. By setting $\vec{v}(t) = L^T \vec{z}(t)$, we can put this system into feedback and set the eigenvalues as we please.

Finally, we can now write down the closed loop system with feedback and observer. The input is $\vec{u}(t) = K\vec{\hat{x}}(t) = K(\vec{x}(t) + \vec{e}(t))$:

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t) = A\vec{x}(t) + BK(\vec{x}(t) + \vec{e}(t)) = (A+BK)\vec{x}(t) + BK\vec{e}(t)$$

Resulting in

$$\begin{bmatrix} \vec{x}(t+1) \\ \vec{e}(t+1) \end{bmatrix} = \begin{bmatrix} A+BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ \vec{e}(t) \end{bmatrix}.$$

In continuous time the math is the same, except for (1) we replace the $\vec{x}(t+1)$ by $\frac{d}{dt}\vec{x}(t)$, and (2) we place the eigenvalues so their real part is less than 0.

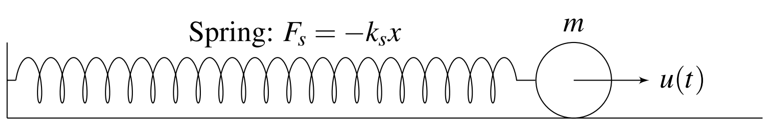
1. Observers

We would like to model a self-propelled mass connected by a string. It is subject to the following forces:

- The spring applies a force $F_s(t) = -k_s x(t)$ (at $x = 0$ it does not apply any force).
- The mass is propelled a force $u(t)$, the latest in perpertuum mobile technology. This is the only input of the system.

The only sensor the system has is a speedometer. That is, it can measure the speed of the system accurately, but it cannot directly measure its position.

Starting at an arbitrary position and speed, we would like to apply the correct inputs required to position the mass at $x = 0$.



(a) Model the system as a linear continuous-time state-space model:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + Bu(t)$$
$$y(t) = C \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

Write down A, B, and C.

(b) Given $m = 1$kg and $k_s = 2$. Plug the values into the system.

(c) Is the system observable?

(d) Write down the equations of the observer system.

(e) Write down the equations governing the estimation error:

$$\vec{e}(t) = \begin{bmatrix} e_0 \\ e_1 \end{bmatrix}.$$

(f) Compute an output-feedback L that places both the eigenvalues of the system governing the estimation error at -2.

(g) Write out the dual system of the original system.

(h) Is the dual system controllable?

(i) Write out the observer closed loop-system using L.

(j) Is the original system stable?

(k) Is the original system controllable?

(l) Derive a state-feedback F that places both the eigenvalues of the closed-loop system at -1.

(m) Write down the equations for the closed loop system, including the feedback and observer.

Contributors:

- Saavan Patel.
- Baruch Sterin.