EECS 16B Designing Information Devices and Systems II
Fall 2016 Murat Arcak and Michel Maharbiz Discussion 9B

Notes

Singular Value Decomposition

The SVD is a useful way to characterize a matrix. Let A be a matrix from $\mathbb{C}^n$ to $\mathbb{C}^m$. The SVD of matrix A is the following decomposition.

$$A = U\Sigma V^*$$

Here, V^* refers to the adjoint of V. For this class, this is simply $\overline{V^T}$. In words, V^* is the transpose of V with all the elements complex conjugated.

The three matrices have nice properties.

- U is an $[m \times m]$ matrix whose columns consist of orthonormal vectors that form a basis for $\mathbb{C}^m$. Consequently,
$$U^*U = I$$
- V is an $[n \times n]$ matrix whose columns consist of orthonormal vectors that form a basis for $\mathbb{C}^n$. Consequently,
$$V^*V = I$$
- U is useful for characterizing the column space of A and V is useful for characterizing the row space of A.
- Σ is a $[m \times n]$ whose diagonal entries are the singular values of A arranged in descending order. Singular values are obtained from the square roots of the eigenvalues of A^*A (or, identically, AA^*) and can be zero.

We calculate the SVD as follows.

(a) Pick AA^* or A^*A.

(b) Let λ_i denote the i^{th} eigenvalue of the chosen matrix, σ_i denote the i^{th} singular value of A and Λ be the diagonal matrix consisting of the eigenvalues of the chosen matrix.

  - If using A^*A, calculate its eigenvalue decomposition to assign Σ and V
$$A^*A = V\Lambda V^* \text{ with } \sigma_i = \sqrt{\lambda_i}$$
  - If using AA^*, calculate its eigenvalue decomposition to assign Σ and U
$$AA^* = U\Lambda U^* \text{ with } \sigma_i = \sqrt{\lambda_i}$$

Note that the non-diagonal entries of Σ will be zero. The various singular values $\{\sigma_i\}$ calculated will be on the diagonal.

(c) Regardless of which matrix picked in step (1), calculate the remaining matrix (either U or V) using the following to ensure everything is sign consistent.

$$Av_i = \sigma_i u_i$$

If any singular value is 0 or you seem to have run out of vectors to completely construct the U or V matrix, complete the basis (or columns of the appropriate matrix) using gram-schmidt. Remember to orthonomalize afterwards.

Questions

1. SVD and Fundamental Subspaces

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

(a) Find the SVD of A.

(b) Find the rank of A.

(c) Find a basis for the kernel (or nullspace) of A.

(d) Find a basis for the range (or columnspace) of A.

2. SVD and Induced 2-Norm

(a) Show that if U is an orthogonal matrix then for any $\vec{x}$

$$||U\vec{x}|| = ||\vec{x}||.$$

(b) Find the maximum

$$\max_{\{\vec{x}:||\vec{x}||=1\}} ||A\vec{x}||$$

in terms of the singular values of A.

(c) Find the $\vec{x}$ that maximizes the expression above.

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